A.M. ≥ G.M.   (Backward Induction)

For n non-negative numbers $a_1, a_2, \ldots, a_n$,

let P(n) be the proposition: $\frac{1}{n}\sum_{i=1}^{n} a_i \geq \sqrt[n]{a_1 a_2 \ldots a_n}$

(a) Show that P(1), P(2) are true.

(b) Show that P(r) is true ⇒ P(2r) is true $\forall r \in \mathbf{N}$.

(c) Show that $P(2^k)$ are true $\forall k \in \mathbf{N}$.

(d) Show that P(n) is true ⇒ P(n – 1) is true $\forall n \in \mathbf{N}$.

(e) Show that P(n) is true $\forall n \in \mathbf{N}$.

Solution

(a) For P(1), L.H.S. = R.H.S. = a_1, P(2) ⇔ $\left(\sqrt{a_1} - \sqrt{a_2}\right)^2 \geq 0$

∴ P(1) and P(2) are true.

(b) Assume P(r) is true for some $r \in \mathbf{N}$.

i.e. $\frac{1}{r}\sum_{i=1}^{r} a_i \geq \sqrt[r]{a_1 a_2 \ldots a_r}$ (1)

For P(2r), $\frac{1}{2r}\sum_{i=1}^{2r} a_i = \frac{1}{2}\left(\frac{1}{r}\sum_{i=1}^{r} a_i + \frac{1}{r}\sum_{i=r+1}^{2r} a_i\right) \geq \sqrt[2]{\left(\frac{1}{r}\sum_{i=1}^{r} a_i\right)\left(\frac{1}{r}\sum_{i=r+1}^{2r} a_i\right)}$

$\geq \sqrt[2]{\sqrt[r]{a_1 a_2 \ldots a_r}\sqrt[r]{a_{r+1} a_{r+2} \ldots a_{2r}}} = \sqrt[2r]{a_1 a_2 \ldots a_r a_{r+1} a_{r+2} \ldots a_{2r}}$

∴ P(2r) is true.

(c) $P(2^1)$ is true ⇒ P(2) is true by (a)

$P(2^k)$ is true ⇒ $P(2 \times (2^k)) = P(2^{k+1})$ is true by (b).

∴ $P(2^k)$ is true $\forall k \in \mathbf{N}$.

(d) Assume P(n) is true. i.e. $\frac{1}{n}\sum_{i=1}^{n} a_i \geq \sqrt[n]{a_1 a_2 \ldots a_n}$ (2)

For P(n – 1), Put $a_n = \frac{a_1 + \ldots + a_{n-1}}{n-1}$

Then $\dfrac{a_1+\dots+a_{n-1}}{n-1} = \dfrac{a_1+\dots+a_n}{n} \geq (a_1a_2\dots a_n)^{1/n}$, by (2).

$$= \left(a_1a_2\dots a_{n-1}\frac{a_1+\dots+a_{n-1}}{n-1}\right)^{1/n} = (a_1a_2\dots a_{n-1})^{1/n}\left(\frac{a_1+\dots+a_{n-1}}{n-1}\right)^{1/n}$$

$$\therefore \quad \left(\frac{a_1+\dots+a_{n-1}}{n-1}\right)^{1-\frac{1}{n}} \geq (a_1a_2\dots a_{n-1})^{1/n}$$

$$\therefore \quad \frac{a_1+\dots+a_{n-1}}{n-1} \geq (a_1a_2\dots a_{n-1})^{1/(n-1)}$$

(e) $\forall n \in \mathbf{N}$ $\exists k, r \in \mathbf{N}$ $\ni$ $n = 2^k - r$

By (c), $P(2^k)$ is true.

But $P(2^k)$ is true $\Rightarrow$ $P(2^k - 1)$ is true

$\Rightarrow$ $P(2^k - 2)$ is true

$\Rightarrow$ (after finite no. of steps)

$\Rightarrow$ $P(2^k - r) = P(n)$ is true.