Some proofs of A.M. ≥ G.M.

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1. (a) Let k be any positive integer and x any positive number.
Put $x^0 = 1$. Show that:

(i) $kx^k < x^{k-1} + \dots + x + 1$ if $x < 1$

(ii) $kx^k > x^{k-1} + \dots + x + 1$ if $x > 1$

Hence show that $kx^{k+1} \geq (k + 1)x^k$
and determine the value of x for which equality holds.

(b) Let $x_1, \dots, x_k, x_{k+1}$ be any k + 1 (k ≥ 1) positive numbers.

By putting $x^{k(k+1)} = \dfrac{x_1}{x_{k+1}}\dfrac{x_2}{x_{k+1}}\dots\dfrac{x_k}{x_{k+1}}$ and using **(a)** , show that:

$$k(x_1x_2\dots x_k)^{1/k} + x_{k+1} \geq (k+1)(x_1x_2\dots x_kx_{k+1})^{1/k+1}$$

and the equality holds if and only if $\dfrac{x_1}{x_{k+1}}\dfrac{x_2}{x_{k+1}}\dots\dfrac{x_k}{x_{k+1}} = 1$

(c) Using **(b)** or otherwise, show that, for any n (n ≥ 1) positive numbers

$x_1, x_2, \dots, x_n$, $\dfrac{x_1 + x_2 + \dots + x_n}{n} \geq (x_1x_2\dots x_n)^{1/n}$

and that the equality holds if and only if $x_1 = x_2 = \dots = x_n$.

2. (a) Show that, if p > 0, q > 0 and n is a positive integer, then

$p^{n+1} - (n + 1)pq^n + nq^{n+1} = p(p^n - q^n) + nq^n (q - p) > 0$ unless p = q.

(b) Hence, or otherwise, show that, if $a_i > 0$ for all i, and

$$A_n = \frac{1}{n}(a_1 + a_2 + \dots + a_n), \quad G_n = (a_1a_2\dots a_n)^{1/n}$$

then $n(A_n - G_n) < (n + 1)(A_{n+1} - G_{n+1})$

(c) Deduce that $A_n > G_n$ unless $a_i = a$ for $i = 1, 2, \dots, n$.

3. (a) Show that for two positive numbers a, b , $a^n + b^n \geq a^{n-1}b + b^{n-1}a$
and that equality holds if and only if a = b.

(b) Let $a_1, a_2, \dots, a_n$ be positive numbers. Using (a), show that:

$$(n-1)(a_1^n + a_2^n + \dots + a_n^n) \geq a_1(a_2^{n-1} + \dots + a_n^{n-1}) + a_2(a_1^{n-1} + a_3^{n-1} + \dots + a_n^{n-1}) + \dots + a_n(a_1^{n-1} + a_2^{n-1} + \dots + a_{n-1}^{n-1})$$

and that equality holds if and only if $a_1 = a_2 = \dots = a_n$.

(c) Using **(b)** and mathematical induction, show that:

$$a_1^n + a_2^n + \dots + a_n^n \geq n\, a_1 a_2 \dots a_n,$$

and hence show that A.M. ≥ G..M.

4. **(a)** Show that $\frac{x_1^2 + x_2^2}{2} \geq x_1 x_2$, where $x_1, x_2 \geq 0$.

(b) Show that $x_1^k + x_2^k \geq x_1 x_2^{k-1} + x_1^{k-1} x_2$, where $k \in \mathbb{N}$.

(c) Use **(b)** to show that for $x_1, x_2, \ldots, x_k \geq 0$

$$(k-1)\left(x_1^k + x_2^k + \ldots + x_k^k\right) \geq x_1\left(x_2^{k-1} + x_3^{k-1} + \ldots + x_k^{k-1}\right) + x_2\left(x_1^{k-1} + x_3^{k-1} + \ldots + x_k^{k-1}\right) + \ldots$$
$$+ x_k\left(x_1^{k-1} + x_2^{k-1} + \ldots + x_{k-1}^{k-1}\right) .$$

(d) Prove that $\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{2} \geq x_1 x_2 \ldots x_n$, where $x_i \geq 0$.

(e) Prove that $\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 a_2 \ldots a_n)^{1/n}$, where $a_i \geq 0$.

5. **(a)** Show that if $x < 1 < y$, then $x + y - xy - 1 > 0$.

Hence prove that for any $n+1$ $(n \geq 1)$ real numbers $x_1, \ldots, x_{n+1}$,
if $x_1 < 1 < x_{n+1}$ and $x_1 x_{n+1} + x_2 + \ldots + x_n \geq n$, then $x_1 + x_2 + \ldots + x_n + x_{n+1} \geq n+1$.

(b) Using **(a)** and mathematical induction prove that for any n $(n \geq 1)$ positive real numbers $x_1, \ldots, x_n$, if $x_1 x_2 \ldots x_n = 1$, then $x_1 + x_2 + \ldots + x_n \geq n$.
Hence, or otherwise show that for any n positive real numbers, $a_1, \ldots, a_n$,

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n} .$$

6. **(a)** For any non-negative number x and for any integer $k > 1$, prove that

$$x^k + k - 1 \geq kx \quad \ldots\ (*)$$

When does the equality holds ?

(b) Let n be an integer greater than 1 , $\{a_1, \ldots, a_n\}$ be a set of positive numbers.

For $m = 1, 2, \ldots, n$, let $A_m = \frac{1}{m}\sum_{i=1}^{m} a_i$, $G_m = \left(\prod_{i=1}^{m} a_i\right)^{1/m}$.

(i) Show that, for $m = 2, \ldots, n$, $\left(\frac{G_m}{G_{m-1}}\right)^m = \frac{mA_m - (m-1)A_{m-1}}{G_{m-1}}$ …. (**)

(ii) Making use of (*) and (**) , or otherwise, prove that

$$A_m - G_m \geq \frac{m-1}{m}\left(A_{m-1} - G_{m-1}\right) \quad \text{for } m = 2, 3, \ldots, n .$$

(iii) Deduce that $A_n \geq G_n$ and show that the equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

7. If $f(x) = f(\theta) + (x-\theta) f'(\theta) + \dfrac{(x-\theta)^2}{2!} f''(\theta + m(x-\theta))$, that is, the mean value theorem holds, where $0 < m < 1$ and if $f(x) \geq 0$, prove that

$$\frac{f(x_1)+f(x_2)+....+f(x_n)}{n} \geq f\left(\frac{x_1+x_2+....+x_n}{n}\right)$$

and hence show that if $f(x) = -\ln x$, then $\dfrac{x_1 + x_2 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 .... x_n}$

8. **(a)** Prove by induction that if $x > -1$, then $(1+x)^n \geq 1 + nx \quad \forall n \in \mathbb{N}$.

(b) If $a_1, a_2, ..., a_n$ are positive numbers, let $b_n = \left(\dfrac{a_1 + a_2 + ... + a_n}{n}\right)^n$ and $c_n = a_1 a_2 .... a_n$,

Prove that if $n > 1$, $\dfrac{b_n}{b_{n-1}} \geq a_n$ and deduce that $\dfrac{b_n}{c_n} \geq \dfrac{b_{n-1}}{c_{n-1}} \geq 1$.

(c) Hence show that $\dfrac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 .... a_n}$ and show that the equality holds if and only if $a_1 = a_2 = ... = a_n$.

9. For positive numbers , $a_1, a_2, ..., a_n$ let

$$G(a_1, ..., a_n) = \sqrt[n]{a_1 a_2 .... a_n} \quad \text{(G.M.)} \quad \text{and} \quad A(a_1, ..., a_n) = \frac{a_1 + a_2 + ... + a_n}{n} \quad \text{(A.M.)}$$

(a) Show that if $a_1 = a_2 = ... = a_n = a$, then $G(a_1, ..., a_n) = A(a_1, ..., a_n) = a$.

(b) Show that if $a_1, a_2, ..., a_n$ are not all equal, then there is one a_i which is greater than $G(a_1, ..., a_n)$ and there is one a_j which is less than $G(a_1, ..., a_n)$.

(c) Let $a_1 > G(a_1, ..., a_n) > a_n$ and let $a_1' = G(a_1, ..., a_n)$, $a_2' = a_2, ..., a_{n-1}' = a_{n-1}$ and $a_n' = \dfrac{a_1 a_n}{G(a_1, ..., a_n)}$, show that $G(a_1', ..., a_n') = G(a_1, ..., a_n)$ and $A(a_1', ..., a_n') < A(a_1, ..., a_n)$.

(d) Using the results above, show that $G(a_1, ..., a_n) \leq A(a_1, ..., a_n)$ and that the equality holds if and only if $a_1 = a_2 = ... = a_n$.

10. **(a)** Show that for $a, b > 0$, $(a+b)^n \geq a^n + na^{n-1}b$.

(b) Show that $\left[\dfrac{1}{n}\sum_{i=1}^{n} a_i\right]^n = \left[\dfrac{1}{n-1}\sum_{i=1}^{n-1} a_i + \dfrac{1}{n}\left(a_n - \dfrac{1}{n-1}\sum_{i=1}^{n-1} a_i\right)\right]^n$

(c) For $a_1 \geq a_2 \geq ... \geq a_n$, show that $\dfrac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 .... a_n}$.

11. Let G_n and A_n denote the geometric mean and arithmetic mean of $a_1,\ldots,a_n$ respectively .

(a) Let $a_1, a_2\ldots, a_{r+1}$ be positive numbers such that $a_1 \le a_2 \le \ldots \le a_{r+1}$.

(i) Prove that $a_1 \le G_r \le a_r$, $a_1 \le A_r \le a_r$ and that the equality holds iff $a_1 = a_2 = \ldots = a_r$. Deduce that $A_r = a_{r+1}$ iff $a_1 = a_2 = \ldots = a_r = a_{r+1}$.

(ii) Show that when k is a positive integer and $x > 0$, $(1+x)^k > 1+kx$.

Also show that $(A_{r+1})^{r+1} = \left(A_r + \dfrac{a_{r+1} - A_r}{r+1}\right)^{r+1}$.

Deduce that if $G_r \le A_r$ and $A_r \ne a_{r+1}$, then $G_{r+1} < A_{r+1}$.

(b) Using **(a)**, or otherwise, show that, for any n $(n \ge 1)$ positive numbers $a_1, a_2,\ldots, a_n$, $G_n \le A_n$ and that the equality holds iff $a_1 = a_2 = \ldots = a_r$.

12. If $a_1,\ldots, a_n$ are positive numbers ,show that the function

$f(x) = a_1 + \ldots + a_n + x - n(a_1\ldots.a_n x)^{\frac{1}{n+1}}$ has the minimum value $a_1 + \ldots + a_n - n(a_1\ldots.a_n)^{\frac{1}{n}}$

and that this value is taken when $x = (a_1 a_2\ldots.a_n)^{\frac{1}{n}}$. Deduce A.M. ≥ G.M.

13. Let $f(x_1, x_2,\ldots., x_n)$ be a function of n variables $x_1, x_2,\ldots., x_n$ and that $\sum f(x_1, x_2,\ldots., x_n)$ be the sum of $n!$ quantities obtained from all possible permutations of $x_1, x_2,\ldots., x_n$ in $f(x_1 x_2\ldots.x_n)$, e.g. $\sum x_1 x_2\ldots.x_n = n! x_1 x_2\ldots.x_n$.

(a) Let $\varphi_n = \varphi(x_1, x_2,\ldots., x_n) = \dfrac{x_1^{\ n} + x_2^{\ n} + \ldots. + x_n^{\ n}}{n} - x_1 x_2\ldots.x_n$.

Show that $n!\varphi_n = \sum x_1^{\ n} - \sum x_1 x_2\ldots.x_n$

(b) Let $\varphi_1 = \sum (x_1^{\ n-1} - x_2^{\ n-1})(x_1 - x_2)$, $\varphi_2 = \sum (x_1^{\ n-2} - x_2^{\ n-2})(x_1 - x_2)x_3$,

$\varphi_3 = \sum (x_1^{\ n-3} - x_2^{\ n-3})(x_1 - x_2)x_3 x_4$, ..., $\varphi_{n-1} = \sum (x_1 - x_2)(x_1 - x_2)x_3 x_4\ldots x_n$,

Show that $\varphi_n = \dfrac{1}{2(n!)}(\varphi_1 + \varphi_2 + \ldots + \varphi_{n-1})$.

(c) Let $x_1, x_2,\ldots., x_n$ be non-negative numbers. Show that $\varphi_n \ge 0$ and show that the equality holds if and only if $x_1 = x_2 = \ldots = x_n$. Hence show that A.M. ≥ G.M.

14. **(a)** Prove that for any positive integer n $(n > 1)$ and any positive number x,

$$\left(1+\frac{x}{n}\right)^{n} > \left(1+\frac{x}{n-1}\right)^{n-1}.$$

(b) For any n $(n > 1)$ positive numbers $a_1, a_2, \ldots, a_n$, not all equal, let $a_m = \min\{a_1, a_2, \ldots, a_n\}$

Use **(a)**, show that $\left[\frac{1}{n}\sum_{r=1}^{n} a_r\right]^{n} > a_m\left[\frac{1}{n-1}\sum_{\substack{r=1\\ r\neq m}}^{n} a_r\right]^{n-1}$.

(c) Hence, using induction or otherwise, prove that, for any n positive numbers $p_1, p_2, \ldots, p_n$, not all equal, $\left[\frac{1}{n}\sum_{r=1}^{n} p_r\right]^{n} > \prod_{r=1}^{n} p_r$.

15. **(a)** Let $f(x) = \ln x - x + 1$, $x > 0$. Use the fact that $\frac{d}{dx}\ln x = \frac{1}{x}$,

prove that $f(x) \le 0$, for all $x > 0$.

(b) Let A be the arithmetic mean and G be the geometric mean of n positive real numbers $a_1, a_2, \ldots, a_n$. By putting $x = \frac{a_i}{A}$ in (a), prove that $A \ge G$.

16. **(a)** Given a function $f(x) = e^{x-1} - x$ where x is a real number. Find the minimum value of f(x).

(b) Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be positive numbers.

(i) Show that $e^{\left\{\left(\sum_{i=1}^{n}\frac{a_i}{b_i}\right)-n\right\}} \ge \prod_{i=1}^{n}\frac{a_i}{b_i}$.

(ii) Hence, or otherwise, show that if $\sum_{i=1}^{n}\frac{a_i}{b_i} \le n$, then $\prod_{i=1}^{n} a_i \le \prod_{i=1}^{n} b_i$

(c) By using the result in **(b)**, prove that $AM \ge GM$ for any n positive numbers $a_1, a_2, \cdots, a_n$.

Solution

1. **(a)** $kx^k - \left(x^{k-1} + \ldots + x + 1\right) = \left(x^k - x^{k-1}\right) + \left(x^k - x^{k-2}\right) + \ldots + \left(x^k - x\right) + \left(x^k - 1\right)$

$= (x-1)\left(kx^{k-1} + (k-1)x^{k-2} + \ldots + 2x + 1\right) \begin{cases} < 0 & , x < 1 \\ > 0 & , x > 1 \end{cases}$ (since $x > 0$)

Obviously,

$(x-1)^2\left(kx^{k-1} + (k-1)x^{k-2} + \ldots + 2x + 1\right) \ge 0$ (with equality holds iff $x = 1$)

$\Leftrightarrow (x-1)\left(kx^k - \left(x^{k-1} + \ldots + x + 1\right)\right) \ge 0 \Leftrightarrow (x-1)kx^k - (x-1)\left(x^{k-1} + \ldots + x + 1\right) \ge 0$

$\Leftrightarrow kx^{k+1} - kx^k - \left(x^k - 1\right) \ge 0 \quad \Leftrightarrow kx^{k+1} + 1 \ge (k+1)x^k$

(b) Put $x^{k(k+1)} = \frac{x_1}{x_{k+1}}\frac{x_2}{x_{k+1}}\ldots\frac{x_k}{x_{k+1}}$, $x^{k+1} = \frac{\left(x_1x_2\ldots x_k\right)^{1/k}}{x_{k+1}}$, $x^k = \frac{\left(x_1x_2\ldots x_k\right)^{1/(k+1)}}{x_{k+1}^{k/(k+1)}}$

By **(a)**, $k\frac{\left(x_1x_2\ldots x_k\right)^{1/k}}{x_{k+1}} + 1 \ge (k+1)\frac{\left(x_1x_2\ldots x_k\right)^{1/(k+1)}}{x_{k+1}^{k/(k+1)}}$

$k\left(x_1x_2\ldots x_k\right)^{1/k} + x_{k+1} \ge (k+1)\left(x_1x_2\ldots x_k\right)^{1/(k+1)}x_{k+1}^{1-k/(k+1)} = (k+1)\left(x_1x_2\ldots x_kx_{k+1}\right)^{1/(k+1)}$

Equality holds $\Leftrightarrow x^{k(k+1)} = 1 \Leftrightarrow \frac{x_1}{x_{k+1}}\frac{x_2}{x_{k+1}}\ldots\frac{x_k}{x_{k+1}} = 1$

(c) Use induction. The assertion is obviously true for $n = 1$.

Assume that the assertion is true for n = k. i.e. $x_1 + x_2 + \ldots + x_k \ge k\left(x_1x_2\ldots x_k\right)^{1/k}$.

For $n = k + 1$, $x_1 + x_2 + \ldots + x_k + x_{k+1} \ge k\left(x_1x_2\ldots x_k\right)^{1/k} + x_{k+1}$ (inductive hypothesis)

$\ge (k+1)\left(x_1x_2\ldots x_kx_{k+1}\right)^{1/(k+1)}$ and the proof completes.

Equality holds $\Leftrightarrow \frac{x_1}{x_{k+1}}\frac{x_2}{x_{k+1}}\ldots\frac{x_k}{x_{k+1}} = 1, \quad \forall k = 1,2,\ldots,(n-1)$

$\Leftrightarrow x_1 = x_2 = \ldots = x_n$.

6. **(a)** Let $f(x) = x^k + k - 1 - kx$

$f'(x) = kx^{k-1} - k = k\left(x^{k-1} - 1\right) = k(x-1)\left(x^{k-2} + x^{k-3} + \ldots + x + 1\right)$

For turning point, $f'(x) = 0$. Since $x > 0$, there is one root, i.e. $x = 1$.

$f''(x) = k(k-1)x^{k-2}$, $f''(1) = k(k-1) > 0$, since k is an integer > 1.

$\therefore$ f(x) is a minimum when x = 1 . It is an absolute min.

$\therefore$ $f(x) \ge f(1)$. $\forall x > 0$.

$\therefore$ $x^k + k - 1 \ge kx$ and equality holds when x = 1 .

(b) **(i)** $\left(\dfrac{G_m}{G_{m-1}}\right)^m = \left[\dfrac{(a_1a_2....a_m)^{1/m}}{(a_1a_2....a_{m-1})^{1/(m-1)}}\right]^m = \dfrac{a_1a_2....a_m}{(a_1a_2....a_{m-1})^{1+1/(m-1)}} = \dfrac{a_1a_2....a_{m-1}a_m}{a_1a_2....a_{m-1}G_{m-1}}$

$= \dfrac{a_m}{G_{m-1}} = \dfrac{(a_1+a_2+....+a_m)-(a_1+a_2+....+a_{m-1})}{G_{m-1}} = \dfrac{mA_m-(m-1)A_{m-1}}{G_{m-1}}$

(ii) From (*) , $x^m + m - 1 \ge mx$, put $x = \dfrac{G_m}{G_{m-1}}$,

From (**) , $x^m = \dfrac{mA_m-(m-1)A_{m-1}}{G_{m-1}}$

We therefore have $\dfrac{mA_m-(m-1)A_{m-1}}{G_{m-1}} + m - 1 \ge m\left(\dfrac{G_m}{G_{m-1}}\right)$

Simplify, we get $A_m - G_m \ge \dfrac{m-1}{m}(A_{m-1} - G_{m-1})$.

(iii) We like to use mathematical induction to show that $A_n - G_n \ge 0$.

The proposition is obviously true for n = 1.

We assume that $A_m - G_m \ge 0$ is true for some integer $m \ge 2$.

From **(b) (ii)**, $A_m - G_m \ge \dfrac{m-1}{m}(A_{m-1} - G_{m-1}) \ge 0$, by inductive hypothesis .

$\therefore$ $A_n - G_n \ge 0$ $\forall\, n \in \mathbb{N}$.

Equality holds $\Leftrightarrow \dfrac{G_m}{G_{m-1}} = 1$ and $\left(\dfrac{G_m}{G_{m-1}}\right)^m = \dfrac{a_m}{G_{m-1}} = 1$, by **(b) (ii)**.

$\Leftrightarrow G_1 = G_2 = ... = G_n$ and $a_m = G_m$ $\forall$ m = 2, 3, …, n .

$\Leftrightarrow a_1 = a_2 = ... = a_n$

12. $f'(x) = 1 - (a_1a_2...a_n)(a_1a_2...a_nx)^{\frac{1}{n+1}-1} = 1 - (a_1a_2...a_n)^{\frac{1}{n+1}} x^{-\frac{n}{n+1}}$

For critical values, f'(x) = 0

$\therefore$ $1 - (a_1a_2...a_n)^{\frac{1}{n+1}} x^{-\frac{n}{n+1}} = 0$, $(a_1a_2...a_n)^{\frac{1}{n+1}} x^{-\frac{n}{n+1}} = 1$, $\therefore x = (a_1a_2...a_n)^{\frac{1}{n}}$

Now, $f''(x) = \dfrac{n}{n+1}(a_1a_2...a_n)^{\frac{1}{n+1}} x^{-\frac{2n+1}{n+1}}$

$\therefore$ f ''(x) > 0 when $x = (a_1a_2...a_n)^{\frac{1}{n}}$.

$\therefore$ f(x) attains its min value at $x = (a_1a_2...a_n)^{\frac{1}{n}}$.

And $\min(f(x)) = a_1 + a_2 + ... + a_n + (a_1 ... a_n)^{\frac{1}{n}} - (n+1)\left\{a_1 ... a_n (a_1 a_2 ... a_n)^{\frac{1}{n}}\right\}^{\frac{1}{n+1}}$

$$= a_1 + a_2 + ... + a_n - n(a_1 a_2 ... a_n)^{\frac{1}{n}}$$

To prove A.M. ≥ G..M., use Mathematical Induction,

Let P(n) be the statement : " $a_1 + a_2 + ... + a_n - n(a_1 a_2 ... a_n)^{\frac{1}{n}} \ge 0$

Obviously P(1) is true.

Assume P(k) is true for some $\forall k \in N$.

i.e. $a_1 + a_2 + ... + a_k - n(a_1 a_2 ... a_k)^{\frac{1}{k}} \ge 0$ (*)

For P(k+1), Put $a_n = x$, we get $f(x) = a_1 + a_2 + ... + a_k + x - (k+1)(a_1 a_2 ... a_n x)^{\frac{1}{k+1}}$

By (a) $f(x) \ge$ Min $\{f(x)\}$ = $a_1 + a_2 + ... + a_k - k(a_1 a_2 ... a_n)^{\frac{1}{k}} \ge 0$, by (*)

∴ P(k+1) is also true.

∴ By the Principle of Mathematical Induction, P(n) is true $\forall n \in N$

$$\therefore \quad \frac{a_1 + a_2 + ... + a_n}{n} \ge (a_1 a_2 .... a_n)^{\frac{1}{n}}$$

13. (a) It is easily seen that whatever permutation is used, the function φ_n remains unchanged.

Since $\varphi_n = \dfrac{x_1^{\ n} + x_2^{\ n} + ... + x_n^{\ n}}{n} - x_1 x_2 .... x_n$, summing up, we have

$$n!\varphi_n = \frac{1}{n}\sum\left(x_1^{\ n} + x_2^{\ n} + ... + x_n^{\ n}\right) - \sum x_1 x_2 .... x_n = \frac{1}{n} \times \sum x_1^{\ n} - \sum x_1 x_2 .... x_n$$

$$= \sum x_1^{\ n} - \sum x_1 x_2 .... x_n$$

(b) We get, from definition, $\varphi_1 = 2\sum x_1^{\ n} - 2\sum x_1^{\ n-1} x_2$,

$$\varphi_2 = 2\sum x_1^{\ n-1} x_2 - 2\sum x_1^{\ n-2} x_2 x_3 ,$$
$$\varphi_3 = 2\sum x_1^{\ n-2} x_2 x_3 - 2\sum x_1^{\ n-3} x_2 x_3 x_4 , \quad .....$$
$$\varphi_{n-1} = 2\sum x_1^{\ 2} x_2 x_3 ... x_{n-1} - 2\sum x_1 x_2 x_3 ... x_n$$

Summing up, $\varphi_1 + \varphi_2 + ... + \varphi_{n-1} = 2\sum x_1^{\ n} - 2\sum x_1 x_2 .... x_n$

By **(a)** , $\varphi_n = \dfrac{1}{2(n!)}(\varphi_1 + \varphi_2 + ... + \varphi_{n-1})$.

(c) From the definition of φ_i , $\varphi_1 = \sum (x_1 - x_2)^2 \left(x_1^{\ n-2} + x_1^{\ n-3} x_2 + ... + x_2^{\ n-2}\right) \ge 0$,

$$\varphi_2 = \sum (x_1 - x_2)^2 \left(x_1^{n-3} + x_1^{n-4} x_2 + \ldots + x_2^{n-3}\right) x_3 \geq 0, \quad \ldots\ldots\ldots,$$

$$\varphi_{n-1} = \sum (x_1 - x_2)^2 x_3 x_4 \ldots x_n \geq 0$$

$$\therefore \quad \varphi_n = \frac{1}{2(n!)} (\varphi_1 + \varphi_2 + \ldots + \varphi_{n-1}) \geq 0$$

It is evident that $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ vanish iff $x_1 = x_2 = \ldots = x_n$.
φ_n vanish iff $x_1 = x_2 = \ldots = x_n$.

Putting $a_1 = x_1^n,\ a_2 = x_2^n, \ldots,\ a_n = x_n^n$ in $\varphi_n \geq 0$, we get A.M. ≥ G.M.

Equality holds if $a_1 = a_2 = \ldots = a_n$.

14. **(a)** Let $f(x) = \left(1 + \frac{x}{n}\right)^n \Big/ \left(1 + \frac{x}{n-1}\right)^{n-1} = \frac{(n-1)^{n-1}}{n^n} \times \frac{(n+x)^n}{(n-1+x)^{n-1}}$

$$f'(x) = \frac{(n-1)^{n-1}}{n^n} \times \frac{x(n-1+x)^{n-2}(n+x)^{n-1}}{(n-1+x)^{2(n-1)}} > 0$$, or any positive integer n $(n > 1)$

$\therefore$ f(x) is increasing in (0, ∞) and f(x) > f(0) = 1.

$$\therefore \quad \left(1 + \frac{x}{n}\right)^n > \left(1 + \frac{x}{n-1}\right)^{n-1}$$

(b) Put $x = \frac{\sum_{r=1}^{n} a_r}{a_m} - n$, then $x = \frac{a_1 - a_m}{a_m} + \frac{a_2 - a_m}{a_m} + \ldots + \frac{a_n - a_m}{a_m} > 0$.

Substitute in **(a)**, we get $\left(1 + \frac{\sum_{r=1}^{n} a_r \big/ a_m - n}{n}\right)^n > \left(1 + \frac{\sum_{r=1}^{n} a_r \big/ a_m - n}{n-1}\right)^{n-1}$.

$$\Rightarrow \frac{1}{a_m^n} \left(\frac{\sum_{r=1}^{n} a_r}{n}\right)^n > \frac{1}{a_m^{n-1}} \left(\frac{\sum_{r=1}^{n} a_r - a_m}{n-1}\right)^{n-1} \Rightarrow \left[\frac{1}{n} \sum_{r=1}^{n} a_r\right]^n > a_m \left[\frac{1}{n-1} \sum_{\substack{r=1 \\ r \neq m}}^{n} a_r\right]^{n-1}$$

(c) Let $s = \sum_{r=1}^{n} p_r$.

Use induction. The assertion is easily proved for $n = 2$, for $(p_1 - p_2)^2 > 0$,

where $p_1, p_2 > 0$ and unequal.
Assume that the assertion is true for the case with $(n - 1)$ numbers.

Now, $\left(\frac{s}{n}\right)^n > p_m\left(\frac{s-p_m}{n-1}\right)^{n-1}$, by **(b)** where $p_m = \min\{p_1, p_2, ..., p_n\}$.

$$> p_m \prod_{\substack{r=1 \\ r\neq m}}^{n} p_r \quad \text{, by inductive hypothesis.}$$

$$= \prod_{r=1}^{n} p_r$$

15. **(a)** $f(x) = \ln x - x + 1$, $x > 0$.

$f'(x) = \frac{1}{x} - 1 = 0$, when $x = 1$.

For $0 < x < 1$, $f'(x) > 0$ and for $x > 1$, $f'(x) < 0$.

Therefore $f(x)$ attains its maximum value at $x = 1$.

$\therefore f(x) \le f(1) = \ln 1 + 1 - 1 = 0, \quad \forall x > 0$.

$\therefore f(x) \le 0, \quad \forall x > 0$.

(b) Let $A = \frac{a_1 + a_2 + ... + a_n}{n}$.

Then for $i = 1, 2, \ldots, n$, $f'\left(\frac{a_i}{A}\right) = \ln\frac{a_i}{A} - \frac{a_i}{A} + 1 \le 0$

$\therefore \quad \ln\frac{a_i}{A} \le \frac{a_i}{A} - 1$

$$\sum_{k=1}^{n} \ln\frac{a_i}{A} \le \sum_{k=1}^{n}\left(\frac{a_i}{A} - 1\right)$$

$$\ln\frac{a_1 a_2 ... a_n}{A^n} \le \sum_{k=1}^{n} \frac{a_i}{A} - \sum_{k=1}^{n} 1 = \frac{1}{A}\sum_{k=1}^{n} a_i - n = \frac{1}{A} nA - n = 0$$

$$\therefore \quad \frac{a_1 a_2 ... a_n}{A^n} \le 1 \Rightarrow \sqrt[n]{a_1 a_2 ... a_n} \le A = \frac{a_1 + a_2 + ... + a_n}{n}$$

16. **(a)** $f(x) = e^{x-1} - x \Rightarrow f'(x) = e^{x-1} - 1$

$f'(x) = 0 \quad \Rightarrow e^{x-1} = 1 \quad \Rightarrow x = 1$

For $x < 1$, $f'(x) < 0$.

For $x > 1$, $f'(x) > 0$.

$\therefore$ At $x = 1$, $f(x)$ attains an absolute minimum value of $f(1) = 0$.

(b) **(i)** By **(a)** , $e^{x-1} \ge x \quad \forall x \in \mathbb{R}$.

Put $x = \frac{a_i}{b_i}$ where $i = 1, 2, \ldots, n$.

$$e^{\frac{a_i}{b_i}-1} \geq \frac{a_i}{b_i} \quad \Rightarrow \prod_{i=1}^{n} e^{\frac{a_i}{b_i}-1} \geq \prod_{i=1}^{n} \frac{a_i}{b_i} \quad \Rightarrow e^{\sum_{i=1}^{n}\left(\frac{a_i}{b_i}-1\right)} \geq \prod_{i=1}^{n} \frac{a_i}{b_i} \quad \Rightarrow e^{\left(\sum_{i=1}^{n}\frac{a_i}{b_i}\right)-n} \geq \prod_{i=1}^{n} \frac{a_i}{b_i}$$

(ii) If $\sum_{i=1}^{n} \frac{a_i}{b_i} \leq n$, then $\sum_{i=1}^{n} \frac{a_i}{b_i} - n \leq 0 \Rightarrow e^{\left(\sum_{i=1}^{n}\frac{a_i}{b_i}\right)-n} \leq 1 \Rightarrow \prod_{i=1}^{n} \frac{a_i}{b_i} \leq 1$

Hence $\prod_{i=1}^{n} a_i \leq \prod_{i=1}^{n} b_i$

(c) Put $b_i = \frac{\sum_{i=1}^{n} a_i}{n} \quad \forall i = 1, 2, \ldots, n$.

then $$\sum_{i=1}^{n} \frac{a_i}{b_i} = \sum_{i=1}^{n} \frac{a_i}{\sum_{i=1}^{n} a_i \Big/ n} = n$$

By **(b)** , $$\prod_{i=1}^{n} a_i \leq \prod_{i=1}^{n} \frac{\sum_{i=1}^{n} a_i}{n} = \left(\frac{\sum_{i=1}^{n} a_i}{n}\right)^n$$

Hence $\left(\prod_{i=1}^{n} a_i\right)^{1/n} \leq \frac{\sum_{i=1}^{n} a_i}{n}$ i.e. G..M. $\leq$ A.M.