Series

1. The sum of the first, second and third terms of a geometric progression to the sum of the third, fourth and fifth terms is $4:9$. Find the tenth term of the series, if the sixth term is $15\frac{3}{16}$.

2. Find the sum to n terms of the series : $1+\frac{x}{a}(1+x)+\frac{x^2}{a^2}\left(1+x+x^2\right)+\frac{x^3}{a^3}\left(1+x+x^2+x^3\right)+\ldots$.

3. If s_n denotes the sum of n terms of the series $1+r+r^2+r^3+\ldots \quad (r\neq 1)$, show that

$$\frac{s_1+s_2+\ldots+s_{n-1}}{n}=\frac{n-s_n}{n(1-r)} .$$

4. Find the sum of the first n terms of a series whose nth term is $\frac{2^n-1}{3^{n+1}}$.

5. Sum to n terms : $\left(1+\frac{1}{r}\right)^2+\left(1+\frac{1}{r^3}\right)^2+\left(1+\frac{1}{r^5}\right)^2+\ldots$.

6. The nth term of a certain series is of the form $a + bn + cn^2$, where a, b, c are numbers . If the first three terms are 2, -1, -3 , find the values of a, b and c and the sum of the first n terms .

7. Find the sum of n terms of the series : $1+(1+x)\sin\theta+\left(1+x+x^2\right)\sin^2\theta+\left(1+x+x^2+x^3\right)\sin^3\theta+\ldots$,

and prove that the sum approaches the limit $\frac{1}{(1-\sin\theta)(1-x\sin\theta)}$ as n increases indefinitely, provided that x lies between certain limits. What are these limits ?

8. Find the value of $\sqrt[3]{a\sqrt[4]{b\sqrt[3]{a\sqrt[4]{b\ldots}}}}$ continued to infinity. (This value is assumed to exist.)

9. Prove that the sum of the terms within the nth – bracket of the series : (1) + (3 + 5) + (7 + 9 + 11) + (13 + 15 + 17 + 19) + … is n^3, and that the sum of the terms in the first n brackets is $\frac{1}{4}n^2(n+1)^2$.

10. Let $S_k=1^k+2^k+3^k+\ldots+n^k$.

(i) Prove that $S_1=\frac{n(n+1)}{2}, \quad S_2=\frac{n(n+1)(2n+1)}{6}, \quad S_3=\frac{n^2(n+1)^2}{4}$.

(ii) Prove the following general formula

$$(k+1)S_k+\frac{(k+1)k}{1\times 2}S_{k-1}+\frac{(k+1)k(k-1)}{1\times 2\times 3}S_{k-2}+\ldots+(k+1)S_1+S_0=(n+1)^{k+1}-1$$

(iii) Put $S_k(n)=1^k+2^k+3^k+\ldots+n^k$, prove the formula

$nS_k(n)=S_{k+1}(n)+S_k(n-1)+S_k(n-2)\ldots+S_k(2)+S_k(1)$.

(iv) **(a)** Prove that $S_k=1^k+2^k+3^k+\ldots+n^k=An^{k+1}+Bn^k+Cn^{k-1}+\ldots+Ln$,

i.e. that the sum $S_k(n)$ can be represented as a polynomial of the $(k+1)$ th degree in n with coefficients independent of n and without a constant term .

(b) Show that $A=\frac{1}{k+1}, \qquad B=\frac{1}{2}$.

(v) Show that the following formulas take place

$$S_4 = \frac{n(n+1)(2n+1)\left(3n^2+3n-1\right)}{30}$$

$$S_5 = \frac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12}$$

$$S_6 = \frac{6n^7+21n^6+21n^5-7n^3+n}{42} = \frac{n(n+1)(2n+1)\left[3n^2(n+1)^2-\left(3n^2+3n-1\right)\right]}{42}$$

$$S_7 = \frac{3n^8+12n^7+14n^6-7n^4+2n^2}{24} = \frac{n^2(n+1)^2\left[3n^2(n+1)^2-2\left(2n^2+2n-1\right)\right]}{24}$$

(vi) Prove that the following relations take place

$$S_3 = S_1^{\,2}, \qquad 4S_1^{\,3} = S_3 + 3S_5, \qquad 2S_5 + S_3 = 3S_2^{\,2}, \qquad S_5 + S_7 = 2S_3^{\,2}$$

11. Determine the sums of the following series

(i) $1+4x+9x^2+\ldots+n^2x^{n-1}$

(ii) $1^3+2^3x+3^3x^2+\ldots+n^3x^{n-1}$

(iii) $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots+\frac{2n-1}{2^{n-1}}$

(iv) $1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\ldots+(-1)^{n-1}\frac{2n-1}{2^{n-1}}$

12. Determine the sums of the following series

(i) $1-2+3-4+\ldots+(-1)^{n-1}n$

(ii) $1^2-2^2+3^2-4^2+\ldots+(-1)^{n-1}n^2$

(iii) $1^2-3^2+5^2-7^2+\ldots-(4n-1)^2$

(iv) $2\times1^2+3\times2^2+4\times3^2+\ldots+(n+1)n^2$.

13. Find the sum of n numbers of the form 1, 11, 111, 1111, ….

14. Prove the identity

(i) $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots+\frac{1}{n(n+1)} = 1-\frac{1}{n+1}$

(ii) $\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+\ldots+\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right)$

(iii) $\frac{1}{1\times3\times5}+\frac{2}{3\times5\times7}+\frac{3}{5\times7\times9}+\ldots+\frac{n}{(2n-1)(2n+1)(2n+3)} = \frac{n(n+1)}{2(2n+1)(2n+3)}$

15. Compute the sum $S = \frac{1^4}{1\times3}+\frac{2^4}{3\times5}+\frac{3^4}{5\times7}+\ldots+\frac{n^4}{(2n-1)(2n+1)}$

16. Find the sum of n terms of the series whose nth term is

(i) $n^2(n+1)^2$

(ii) $\left(n^2+5n+4\right)\left(n^2+5n+8\right)$

(iii) $\frac{n^2(n+1)^2}{4n^2-1}$

(iv) $\frac{n^4+2n^3+n^2-1}{n^2+n}$

(v) $\frac{n^3+3n^2+2n+2}{n^2+2n}$

(vi) $\frac{n^4+n^2+1}{n^4+n}$