Transformation

1. Find the 2×2 matrices representing the following transformations:

(a) First an expansion by a factor of 2 in the x-direction, then a reflection in the x-axis.

(b) First a reflection in the line $y = -x$, then a rotation through an angle of 90° anti-clockwisely about the origin.

(c) First a rotation about the origin through an angle of 45° anti-clockwisely, then an expansion by a factor of $\sqrt{2}$ in the x-direction, then finally an expansion by a factor of $2\sqrt{2}$ in the y-direction.

Ans. **(a)** $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$

2. Show that the unit circle centre origin under the transformation of an expansion in the x-direction with a factor of a and an expansion in the y-director with a factor b becomes an ellipse with major axis a and minor axis b.

3. The points on the R^2 plane are transformed by the following equations: $\begin{cases} x' = -x + \sqrt{3}\,y \\ y' = \sqrt{3}\,x + y \end{cases}$.

(a) If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix representing the above transformation, i.e. $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$,

find $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(b) Express the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ found in **(a)** in the form $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

where $k > 0$.

Hence, describe the geometrical meaning of the transformation.

(c) Find the equation of the image of the curve (Γ): $x^2 + 3y^2 - 2\sqrt{3}\,xy - 4\sqrt{3}\,x - 4y = 0$

under the above transformation.

Hence, describe the shape of the curve (Γ).

4. A triangle has vertices $A(-3, -2)$, $B(1, 1)$ and $C(-1, 2)$.

(a) **(i)** Write down, in matrix form, a translation Γ, which maps A to the origin.

(ii) Write down the coordinates of the images B' of B and C' of C under the translation Γ.

(b) **(i)** Find the matrix of the rotation, R, about the origin which will map B' to a point on the positive x-axis.

(ii) Find the coordinates of the images B" of B' and C" of C' under the rotation R.

(iii) Hence, calculate the area of ΔABC.

Ans. (a) (ii) B' = $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, C' = $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ **(b)** **(i)** $\frac{1}{5}\begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ **(iii)** 5

5. In each of the following, find the equation of the image of the curve (Γ) under the transformation specified.

(a) (Γ) : $5x^2 + 7y^2 - 2\sqrt{3}\,xy - 4 = 0$ [a rotation through an angle of $\frac{5\pi}{6}$ about the origin.]

(b) (Γ) : $y^2 - x^2 - 2\sqrt{3}\,xy - 2 = 0$ [a reflection in the line through the origin making an angle of $\frac{\pi}{3}$ with the positive x-axis.]

(c) (Γ) : $x^2 + y^2 - 4x = 0$ [a reflection in the line $y = x$ followed by a shear $\begin{pmatrix} 1 & k \\ c & 1 \end{pmatrix}$ which maps (1 , 1) to (-1 , -3).]

Ans. **(a)** $X^2 + 2Y^2 = 1$ **(b)** $X^2 - Y^2 = 1$ **(c)** $17X^2 + 5Y^2 + 12XY + 112X + 28Y = 0$

6. **(a)** Find a symmetric matrix Q such that the conic equation $5x^2 + 8y^2 - 4xy - 36 = 0$ can be written as $(x \quad y)\,Q\begin{pmatrix} x \\ y \end{pmatrix} - 36 = 0$. **Ans.** $\begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}$

(b) Let P be the matrix of rotation through an acute angle θ about the origin.

(i) Express P in terms of θ.

(ii) Show that $$P\,Q\,P^{-1} = \begin{pmatrix} 5 + 3\sin^2\theta + 2\sin 2\theta & -\frac{3}{2}\sin 2\theta - 2\cos 2\theta \\ -\frac{3}{2}\sin 2\theta - 2\cos 2\theta & 5 + 3\cos^2\theta - 2\sin 2\theta \end{pmatrix}.$$

(iii) Hence, find the value of $\tan\theta$ if $P\,Q\,P^{-1}$ is a diagonal matrix. **Ans.** $\tan\theta = 2$

(c) With the value of θ found in **(b)**, find the equation of the image of the conic in (a) under the rotation represented by P. **Ans.** $9X^2 + 4Y^2 = 36$

7. Let T be the translation which maps the origin to the point (α , β) and Q be the reflection about the line $x - \sqrt{3}\,y = 0$. Let (Γ) : $x^2 - y^2 - 2\sqrt{3}\,xy - 2x + 2\sqrt{3}\,y + 3 = 0$.

(a) If (x' , y') is the image of (x , y) under T,

(i) find the equation of the image of (Γ) under T,

($(x' - \alpha)^2 - 2\sqrt{3}(x' - \alpha)(y' - \beta) - (y' - \beta)^2 - 2(x' - \alpha) + 2\sqrt{3}(y' - \beta) + 3 = 0$)

(ii) hence, determine the values of α and β so that the equation in **(a) (i)** has no x' and y' terms.

(b) **(i)** Find the matrix of the reflection Q. **Ans.** $\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

(ii) If (X , Y) is the image of (x' , y') under Q , show that $\begin{cases} x' = \dfrac{X + \sqrt{3}\,Y}{2} \\ y' = \dfrac{\sqrt{3}\,X - Y}{2} \end{cases}$.

(c) Hence, find the equation of the image of the curve (Γ) under the translation T followed by the reflection Q.