# Quadratics

**Quadratic equation**

(1) General form : $ax^2 + bx + c = 0$, $a \neq 0$, a, b, c are real numbers.

(2) Factorized form: $(x - \alpha)(x - \beta) = 0$, $\alpha, \beta$ are roots of the quadratic equation.

(3) Expanded form: $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2$ – (sum of roots)x + product of roots = 0

**Quadratic equation formula**

(1) $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

(2) $$ax^2 + bx + c = 0 \Rightarrow c\left(\frac{1}{x}\right)^2 + b\left(\frac{1}{x}\right) + a = 0 \Rightarrow \frac{1}{x} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c} \Rightarrow x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}$$

(3) $\frac{1}{\alpha}, \frac{1}{\beta}$ are roots of the equation : $cx^2 + bx + a = 0$

**Discriminant**

Define $\Delta = b^2 - 4ac$

(1) $\Delta > 0 \Rightarrow$ 2 unequal real roots.

(2) $\Delta = 0 \Rightarrow$ repeated (equal) roots. $x = -\frac{b}{2a}$

(3) $\Delta < 0 \Rightarrow$ no real roots (two complex conjugate roots)

(4) $\Delta$ is a perfect square and a, b, c are rational numbers (or integers)

$\Rightarrow$ two rational roots.

**Vieta Theorem**

Sum of roots $= \alpha + \beta = -\frac{b}{a}$, Product of roots $= \alpha\beta = \frac{c}{a}$

**Symmetric forms**

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$\alpha - \beta = \pm\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

General formula for getting symmetric forms : $\alpha^n + \beta^n = (\alpha^{n-1} + \beta^{n-1})(\alpha + \beta) - \alpha\beta(\alpha^{n-2} + \beta^{n-2})$, $n \geq 2$.

Sum & Product of roots $\Rightarrow$ Roots $$\alpha, \beta = \frac{(\alpha + \beta) \pm (\alpha - \beta)}{2} = \frac{(\alpha + \beta) \pm \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}}{2}$$

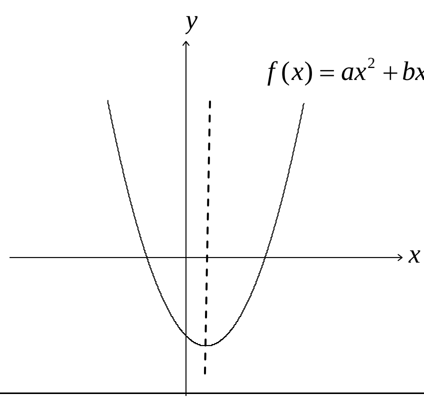
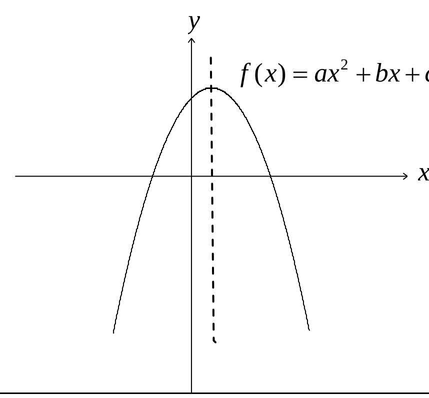
## Quadratic function

$f(x) = ax^2 + bx + c,\quad a \neq 0,\quad$ a, b, c are real numbers. (General form)

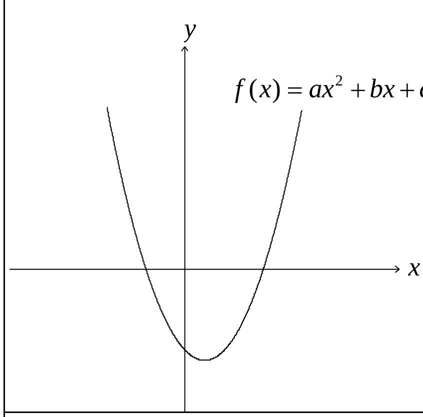
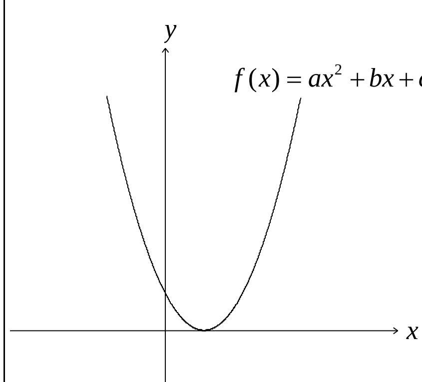
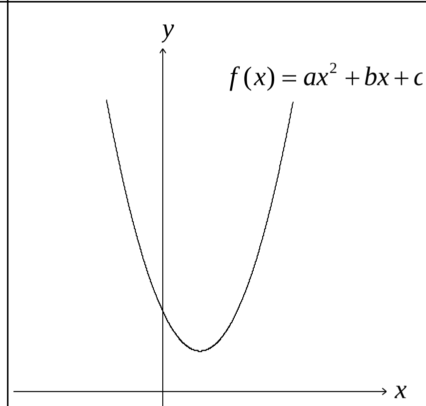
$= a(x - \alpha)(x - \beta) = a\left(x - \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}\right)$ (Factorized form)

$= a[x^2 - (\alpha + \beta)x + \alpha\beta]$ (Expanded form)

$= a\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{\Delta}{4a}$ (Complete square)

| **a > 0** | **a < 0** |
|---|---|
|  |  |
| Concave upwards | Concave downwards |
| Minimum of $y = -\dfrac{\Delta}{4a}$, when $x = -\dfrac{b}{2a}$ | Maximum of $y = -\dfrac{\Delta}{4a}$, when $x = -\dfrac{b}{2a}$ |
| Line of symmetry: $x = -\dfrac{b}{2a}$ | Line of symmetry: $x = -\dfrac{b}{2a}$ |

## Discriminant

| **Δ > 0, a > 0** | **Δ = 0, a > 0** | **Δ < 0, a > 0** |
|---|---|---|
|  |  |  |
| The curve cuts the x-axis at two points. | The curve touches the x-axis at one point. | The curve does not cut the x-axis. |

## Quadratic inequality

Assume $\alpha < \beta$,

**(1)** $(x-\alpha)(x-\beta) > 0 \Rightarrow x < \alpha$ or $x > \beta$

**(2)** $(x-\alpha)(x-\beta) \geq 0 \Rightarrow x \leq \alpha$ or $x \geq \beta$

**(3)** $(x-\alpha)(x-\beta) < 0 \Rightarrow \alpha < x < \beta$

**(4)** $(x-\alpha)(x-\beta) \leq 0 \Rightarrow \alpha \leq x \leq \beta$

| Ineq. \ Solution ($\Delta$) | $\Delta > 0$ | $\Delta = 0$ | $\Delta < 0$ |
|---|---|---|---|
| $ax^2+bx+c>0$, $a>0$ | $x < \frac{-b-\sqrt{\Delta}}{2a}$ or $x > \frac{-b+\sqrt{\Delta}}{2a}$ | $x \neq -\frac{b}{2a}$ | $-\infty < x < \infty$ |
| $ax^2+bx+c<0$, $a>0$ | $\frac{-b-\sqrt{\Delta}}{2a} < x < \frac{-b+\sqrt{\Delta}}{2a}$ | no real solution | no real solution |
| $ax^2+bx+c\geq 0$, $a>0$ | $x \leq \frac{-b-\sqrt{\Delta}}{2a}$ or $x \geq \frac{-b+\sqrt{\Delta}}{2a}$ | $-\infty < x < \infty$ | $-\infty < x < \infty$ |
| $ax^2+bx+c\leq 0$, $a>0$ | $\frac{-b-\sqrt{\Delta}}{2a} \leq x \leq \frac{-b+\sqrt{\Delta}}{2a}$ | $x = -\frac{b}{2a}$ | no real solution |

**Related inequalities**

**(1) Fractional inequality**

$$\frac{x-\alpha}{x-\beta} \geq 0 \Rightarrow \frac{x-\alpha}{x-\beta}(x-\beta)^2 \geq 0 \text{ and } x \neq \beta \Rightarrow (x-\alpha)(x-\beta) \geq 0 \text{ and } x \neq \beta \Rightarrow \underline{\underline{x \leq \alpha \text{ or } x > \beta}}$$

$$\frac{x-\alpha}{x-\beta} \leq 0 \Rightarrow \frac{x-\alpha}{x-\beta}(x-\beta)^2 \leq 0 \text{ and } x \neq \beta \Rightarrow (x-\alpha)(x-\beta) \leq 0 \text{ and } x \neq \beta \Rightarrow \underline{\underline{\alpha \leq x < \beta}}$$

**(2) Inequality with absolute value**

$|x| < a$, $a > 0 \Rightarrow -a < x < a$

$|x| > a$, $a > 0 \Rightarrow x < -a$ or $x > a$

$|x - a| < b$, $b > 0 \Rightarrow a - b < x < a + b$

$|x - a| > b$, $b > 0 \Rightarrow x < a - b$ or $x > a + b$

**(3) Range of rational function of two quadratics**

If x is real, the range of $y = \frac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}$ can be found by:

**(a)** Rearranging terms : $(a_2y - a_1)x^2 + (b_2y - b_1)x + (c_2y - c_1) = 0$

**(b)** x is real $\Rightarrow \Delta \geq 0 \Rightarrow (b_2y - b_1)^2 \geq 4(a_2y - a_1)(c_2y - c_1)$

**(c)** Expand to get a quadratic inequality in y, solve it to get the range of y.

**(4)** The quadratic function $y = f(x) > 0$ for all real x $(a > 0) \Leftrightarrow \Delta$ of $f(x) < 0$

The quadratic function $y = f(x) < 0$ for all real x $(a < 0) \Leftrightarrow \Delta$ of $f(x) > 0$