Ratio, proportion and variation

(Terms are considered to be **positive**, and can be extended to **negative** in some of the statements below.)

1. $a : b = c : d \quad \Leftrightarrow \quad \frac{a}{b} = \frac{c}{d}$

2. **Cross multiplication:** $a : b = c : d \quad \Leftrightarrow \quad ad = bc$

3. $\begin{cases} \frac{a}{b} > 1 \Rightarrow \frac{a}{b} > \frac{a+x}{b+x} \\ \frac{a}{b} < 1 \Rightarrow \frac{a}{b} < \frac{a+x}{b+x} \end{cases}$, a, b, x > 0. $\qquad \begin{cases} \frac{a}{b} > 1 \Rightarrow \frac{a}{b} < \frac{a-x}{b-x} \\ \frac{a}{b} < 1 \Rightarrow \frac{a}{b} > \frac{a-x}{b-x} \end{cases}$, a > x > 0, b > x > 0.

4. $\frac{a}{b} = \frac{c}{d} \Leftrightarrow \frac{a}{c} = \frac{b}{d}$ (Alterando) , $\frac{a}{b} = \frac{c}{d} \Leftrightarrow \frac{b}{a} = \frac{d}{c}$ (Invertendo)

$\frac{a}{b} = \frac{c}{d} \Leftrightarrow \frac{a+b}{b} = \frac{c+d}{d}$ (Compoendo) , $\frac{a}{b} = \frac{c}{d} \Leftrightarrow \frac{a-b}{b} = \frac{c-d}{d}$ (Dividendo)

$\frac{a}{b} = \frac{c}{d} \Leftrightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$ (Compoendo et Dividendo)

5. **Equal Ratios Theorems**

$\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n} = k \quad \Rightarrow \quad k = \frac{a_1 + a_2 + ... + a_n}{b_1 + b_2 + ... + b_n}$, where $b_1 + b_2 + ... + b_n \neq 0$

$\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n} = k \quad \Rightarrow \quad k = \frac{m_1a_1 + m_2a_2 + ... + m_na_n}{m_1b_1 + m_2b_2 + ... + m_nb_n}$, where $m_1b_1 + m_2b_2 + ... + m_nb_n \neq 0$

$\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n} = k \quad \Rightarrow \quad k = \left(\frac{m_1{a_1}^p + m_2{a_2}^p + ... + m_n{a_n}^p}{m_1{b_1}^p + m_2{b_2}^p + ... + m_n{b_n}^p} \right)^{1/p}$, where $m_1{b_1}^p + m_2{b_2}^p + ... + m_n{b_n}^p \neq 0$

6. **Unequal Ratios Theorem**

$\frac{a_1}{b_1} < \frac{a_2}{b_2} < ... < \frac{a_n}{b_n} \quad \Rightarrow \quad \frac{a_1}{b_1} < \frac{a_1 + a_2 + ... + a_n}{b_1 + b_2 + ... + b_n} < \frac{a_n}{b_n}$, where $a_1, a_2, ..., a_n ; b_1, b_2, \ldots, b_n > 0$

7. $a_1 : b_1 : c_1 = a_2 : b_2 : c_2 \quad \Leftrightarrow \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \quad \Leftrightarrow \quad \begin{cases} a_1 = ka_2 \\ b_1 = kb_2 \\ c_1 = kc_2 \end{cases}$

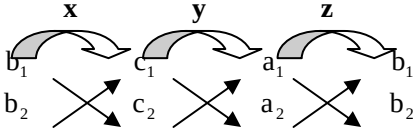
8. **Continued proportion**

$a : b = b : c \quad \Rightarrow \quad a : c = a^2 : b^2 = b^2 ; c^2$

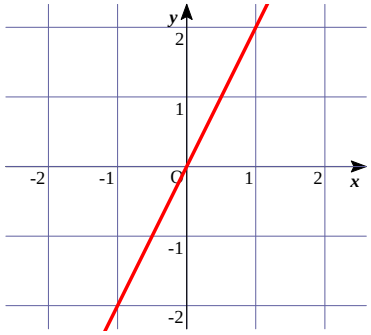
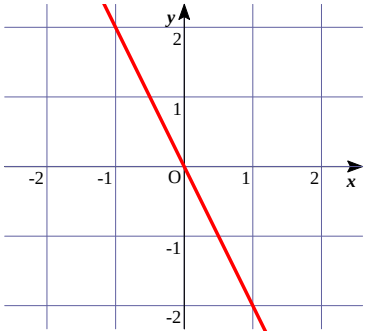
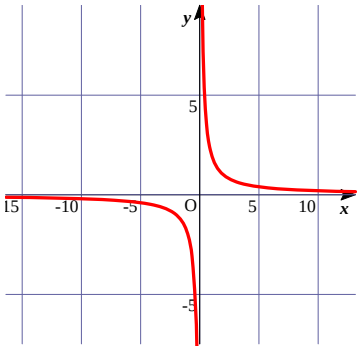
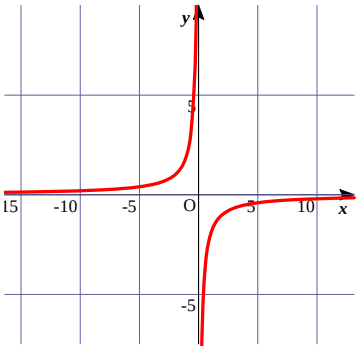
$a : b = b : c = c : d \quad \Rightarrow \quad a : d = a^3 : b^3 = b^3 : c^3 = c^3 : d^3$

9. **Rule of Cross Multiplication**

$\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \end{cases}$



$\Rightarrow \quad \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$

10. Direct variation $x \propto y \Leftrightarrow x = ky \Rightarrow \frac{x_1}{x_2} = \frac{y_1}{y_2}$	**k > 0** 	**k < 0** 
11. Inverse variation $x \propto \frac{1}{y} \Leftrightarrow x = \frac{k}{y} \Rightarrow x_1y_1 = x_2y_2$	**k > 0** 	**k < 0** 

12. Joint variation $x \propto yz \Leftrightarrow x = kyz \Rightarrow \frac{x_1}{x_2} = \frac{y_1z_1}{y_2z_2}$

13. $x \propto y$, when z is kept constant and $x \propto z$ when y is kept constant $\Rightarrow x \propto yz$

14. Partial variation

x is partly constant and partly varies directly as y $\Rightarrow$ $x = k_1 + k_2y$

x is partly varies directly as y and partly varies directly as z $\Rightarrow$ $x = k_1y + k_2z$

x is partly varies directly as y and partly varies inversely as z $\Rightarrow$ $x = k_1y + \frac{k_2}{z}$

15. Transitive law: $x \propto y$ and $y \propto z \Rightarrow x \propto z$

16. $x \propto yz \Leftrightarrow \frac{x}{y} \propto z$

17. $x \propto z$ and $y \propto z \Rightarrow (x \pm y) \propto z$

18. $x \propto z$ and $y \propto z \Rightarrow xy \propto z^2$

19. $a : b = ka : kb = \frac{a}{k} : \frac{b}{k}$, $k \neq 0$.

20. $x \propto y$ and $z \propto w \Rightarrow xz \propto yw$

21. $x \propto y$ and $z \propto w \Rightarrow \frac{x}{z} \propto \frac{y}{w}$

22. $x \propto y \Rightarrow (ax + by) \propto (cx + dy)$, $(a, b), (c, d) \neq (0, 0)$