Trigonometry

Square relations

$\sin^2\theta + \cos^2\theta = 1$ $\quad$ $\tan^2\theta + 1 = \sec^2\theta$ $\quad$ $\cot^2\theta + 1 = \csc^2\theta$

Multiplication

$\sin\theta = \tan\theta\cos\theta$ $\quad$ $\cos\theta = \sin\theta\cot\theta$ $\quad$ $\tan\theta = \sin\theta\sec\theta$

$\cot\theta = \cos\theta\csc\theta$ $\quad$ $\sec\theta = \tan\theta\csc\theta$ $\quad$ $\csc\theta = \sec\theta\cot\theta$

Division

$$\sin\theta = \frac{\cos\theta}{\cot\theta} = \frac{\tan\theta}{\sec\theta} \qquad \cos\theta = \frac{\sin\theta}{\tan\theta} = \frac{\cot\theta}{\csc\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sec\theta}{\csc\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\csc\theta}{\sec\theta} \qquad \sec\theta = \frac{\tan\theta}{\sin\theta} = \frac{\csc\theta}{\cot\theta} \qquad \csc\theta = \frac{\sec\theta}{\tan\theta} = \frac{\cot\theta}{\cos\theta}$$

Complimentary angles

$\sin(90° - \theta) = \cos\theta$ $\quad$ $\cos(90° - \theta) = \sin\theta$ $\quad$ $\tan(90° - \theta) = \cot\theta$

$\cot(90° - \theta) = \tan\theta$ $\quad$ $\sec(90° - \theta) = \csc\theta$ $\quad$ $\csc(90° - \theta) = \sec\theta$

Reciprocal relations

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Negative angles

$\sin(-\theta) = -\sin\theta$ $\quad$ $\cos(-\theta) = \cos\theta$ $\quad$ $\tan(-\theta) = -\tan\theta$

Radian Measure

$$\frac{\theta^\circ}{360} = \frac{\theta^c}{2\pi} = \frac{L}{2\pi r} = \frac{A}{\pi r^2}$$

where $\theta°$ = angle at center in degrees, θ^c = angle at center in radians, r = radius of circle,
L = arc length, A = area of sector.

Sine Law

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R = \frac{abc}{2\Delta} \qquad \sin A : \sin B : \sin C = a : b : c$$

where R is the ***radius of circum-circle*** and Δ is the area of ΔABC.

Projection Law (First Cosine Law)

$a = b\cos C + c\cos B$

$b = c\cos A + a\cos C$

$c = a\cos B + b\cos A$

Cosine Law (Second Cosine Law)

$a^2 = b^2 + c^2 - 2bc \cos A$ $b^2 = c^2 + a^2 - 2ca \cos B$ $c^2 = a^2 + b^2 - 2ab \cos C$

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$ $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Area of triangle

1. $\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C$

2. Heron's formula

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$

Compound angle

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

Double angle

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$ $\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}$

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$ $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

$(\sin x \pm \cos x)^2 = 1 \pm \sin 2x$

Half angle

$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$

$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = 2\cos^2 \frac{x}{2} - 1 = 1 - 2\sin^2 \frac{x}{2}$

$\sin \frac{x}{2} = \pm \sqrt{\dfrac{1 - \cos x}{2}}$ $\cos \frac{x}{2} = \pm \sqrt{\dfrac{1 + \cos x}{2}}$

$\tan \frac{x}{2} = \pm \sqrt{\dfrac{1 - \cos x}{1 + \cos x}} = \dfrac{1 - \cos x}{\sin x} = \dfrac{\sin x}{1 + \cos x}$

Triple angle

$\sin 3x = 3 \sin x - 4 \sin^3 x$

$\cos 3x = 4\cos^3 x - 3\cos x$

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \qquad \cot 3x = \frac{\cot^3 x - 3\cot x}{3\cot^2 x - 1}$$

t-formula

$$t = \tan\frac{x}{2}, \qquad \sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \tan x = \frac{2t}{1-t^2}$$

Products as Sums or Differences

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$-2\sin A\sin B = \cos(A+B) - \cos(A-B)$

Sums or Differences as Products

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Subsidiary Angle Form

$$a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\cos(\theta-\alpha), \text{ where } \tan\alpha = \frac{a}{b}$$

$$a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta-\beta), \text{ where } \tan\beta = \frac{b}{a}$$

$$\sin\theta \pm \cos\theta = \sqrt{2}\sin\left(\theta \pm \frac{\pi}{4}\right) = \pm\sqrt{2}\cos\left(\theta \pm \frac{\pi}{4}\right)$$

General solutions

1. $\sin\theta = k$ $\quad \theta = n\pi + (-1)^n \sin^{-1}k$, where n is an integer.

2. $\cos\theta = k$ $\quad \theta = 2n\pi \pm \cos^{-1}k$, where n is an integer.

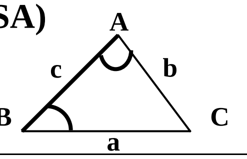
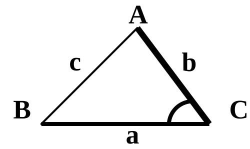
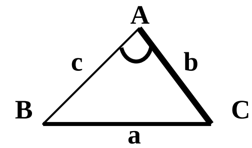
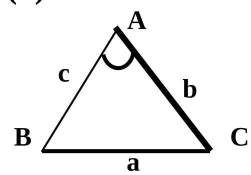
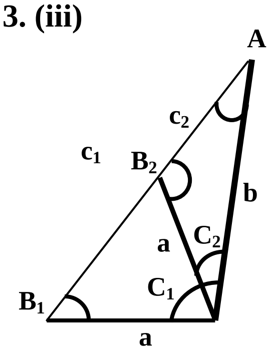
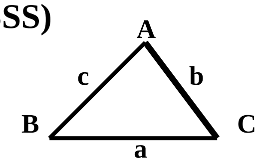
3. $\tan\theta = k$ $\quad \theta = n\pi + \tan^{-1}k$, where n is an integer.

Relation between trigonometric functions

	$\sin\theta = x$	$\cos\theta = x$	$\tan\theta = x$	$\cot\theta = x$	$\sec\theta = x$	$\csc\theta = x$
$\sin\theta$	x	$\pm\sqrt{1-x^2}$	$\pm\dfrac{x}{\sqrt{1+x^2}}$	$\pm\dfrac{1}{\sqrt{1+x^2}}$	$\pm\dfrac{\sqrt{x^2-1}}{x}$	$\dfrac{1}{x}$
$\cos\theta$	$\pm\sqrt{1-x^2}$	x	$\pm\dfrac{1}{\sqrt{1+x^2}}$	$\pm\dfrac{x}{\sqrt{1+x^2}}$	$\dfrac{1}{x}$	$\pm\dfrac{\sqrt{x^2-1}}{x}$
$\tan\theta$	$\pm\dfrac{x}{\sqrt{1-x^2}}$	$\pm\dfrac{\sqrt{1-x^2}}{x}$	x	$\dfrac{1}{x}$	$\pm\sqrt{x^2-1}$	$\pm\dfrac{1}{\sqrt{x^2-1}}$
$\cot\theta$	$\pm\dfrac{\sqrt{1-x^2}}{x}$	$\pm\dfrac{x}{\sqrt{1-x^2}}$	$\dfrac{1}{x}$	x	$\pm\dfrac{1}{\sqrt{x^2-1}}$	$\pm\sqrt{x^2-1}$
$\sec\theta$	$\pm\dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{1}{x}$	$\pm\sqrt{1+x^2}$	$\pm\dfrac{\sqrt{1+x^2}}{x}$	x	$\pm\dfrac{x}{\sqrt{x^2-1}}$
$\csc\theta$	$\dfrac{1}{x}$	$\pm\dfrac{1}{\sqrt{1-x^2}}$	$\pm\dfrac{\sqrt{1+x^2}}{x}$	$\pm\sqrt{1+x^2}$	$\pm\dfrac{x}{\sqrt{x^2-1}}$	x

Some Special Angles

α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$	rad.
0	0	1	0	undefined	$\dfrac{\pi}{2}$
$\dfrac{\pi}{12}$	$\dfrac{\sqrt{6}-\sqrt{2}}{4}$	$\dfrac{\sqrt{6}+\sqrt{2}}{4}$	$2-\sqrt{3}$	$2+\sqrt{3}$	$\dfrac{5\pi}{12}$
$\dfrac{\pi}{10}$	$\dfrac{\sqrt{5}-1}{4}$	$\dfrac{\sqrt{10+2\sqrt{5}}}{4}$	$\dfrac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\dfrac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$	$\dfrac{2\pi}{5}$
$\dfrac{\pi}{8}$	$\dfrac{\sqrt{2-\sqrt{2}}}{2}$	$\dfrac{\sqrt{2+\sqrt{2}}}{2}$	$\sqrt{2}-1$	$\sqrt{2}+1$	$\dfrac{3\pi}{8}$
$\dfrac{\pi}{6}$	$\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{\sqrt{3}}$	$\sqrt{3}$	$\dfrac{\pi}{3}$
$\dfrac{\pi}{5}$	$\dfrac{\sqrt{10-2\sqrt{5}}}{2}$	$\dfrac{\sqrt{5}+1}{4}$	$\dfrac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}+1}$	$\dfrac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$	$\dfrac{3\pi}{10}$
$\dfrac{\pi}{4}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$	1	1	$\dfrac{\pi}{4}$
rad.	$\cos\beta$	$\sin\beta$	$\cot\beta$	$\tan\beta$	β

<table>
<tr><th colspan="3">Solve an oblique triangle</th></tr>
<tr><th>Known conditions</th><th>Procedures for solving</th><th>Diagrams</th></tr>
<tr><td>Two angles & included side : A, B, c</td><td>1. $C = 180° - A - B$
2. $b = \frac{a \sin B}{\sin A}$, $c = \frac{a \sin C}{\sin A}$</td><td>(ASA)
</td></tr>
<tr><td>Two sides & included angle : a, b, C</td><td>1. $c = \sqrt{a^2 + b^2 - 2ab\cos C}$
2. $A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$
or if $a < b$, $A = \sin^{-1}\left(\frac{a \sin C}{c}\right)$
3. $B = 180° - A - C$</td><td>(SAS)
</td></tr>
<tr><td>Two sides & one opposite angle : a, b, A</td><td>1. a > b, there is one unique solution:
$B < 90^\circ$, $B = \sin^{-1}\left(\frac{b \sin A}{a}\right)$
$C = 180° - A - B$, $c = \frac{a \sin C}{\sin A}$
2. a = b,
(i) $A \geq 90°$, there is no solution.
(ii) $A < 90°$, $B = A$, $C = 180° - 2A$, $c = \frac{a \sin C}{\sin A}$
3. a < b, $A \geq 90°$, there is no solution.
$A < 90°$, there are 3 cases:
(i) $a < b \sin A$, there is no solution.
(ii) $a = b \sin A$, there is one solution:
$B = 90°$, $C = 90° - A$
(iii) $a > b \sin A$, there are two solutions:
$B_1 = \sin^{-1}\left(\frac{b \sin A}{a}\right)$, $B_2 = 180° - B_1$
$C_1 = 180° - A - B_1$, $C_2 = 180° - A - B_2$
$c_1 = \frac{a \sin C_1}{\sin A}$ $c_2 = \frac{a \sin C_2}{\sin A}$</td><td>(SSA)
1.

2. (ii)

3. (iii)
</td></tr>
<tr><td>Three sides a, b, c</td><td>For $a > b$, $A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$
$B = \sin^{-1}\left(\frac{b \sin A}{a}\right)$, $C = 180° - A - B$</td><td>(SSS)
</td></tr>
<tr><td>Notes on choosing or rejecting</td><td colspan="2">1. Sine Law may possibly give two solutions. Cosine Law gives one solution.
2. $a + b > c$, $b + c > a$, $c + a > b$.
3. Greater angle, greater opposite sides, and vice versa.
4. There is at most one obtuse angle in a triangle. The smaller angles are acute.</td></tr>
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