**Conics**

Given that the ellipse $\frac{x^{2}}{t+1}+\frac{y^{2}}{t-1}=1$ is tangential to the hyperbola $xy=1$ and $t>1$.

Find the value of t.



**Method 1**

 $xy=1⟹y=\frac{1}{x}⟹\frac{dy}{dx}=-\frac{1}{x^{2}}$

 $\frac{x^{2}}{t+1}+\frac{y^{2}}{t-1}=1⟹\frac{2x}{t+1}+\frac{2y}{t-1}\frac{dy}{dx}=0⟹\frac{dy}{dx}=-\frac{x\left(t-1\right)}{y\left(t+1\right)}$

 Since $\frac{x^{2}}{t+1}+\frac{y^{2}}{t-1}=1$ is tangential to the hyperbola $xy=1$, we have

 $-\frac{1}{x^{2}}=-\frac{x\left(t-1\right)}{y\left(t+1\right)}⟹y=\frac{t-1}{t+1}x^{3} ….(1)$

 Substitute (1) in $xy=1$, $\frac{t-1}{t+1}x^{4}=1⟹x=\pm \sqrt[4]{\frac{t+1}{t-1}}$ , since $t>1$

 Substitute in (1), $y=\frac{t-1}{t+1}\left(\pm \sqrt[4]{\frac{t+1}{t-1}}\right)^{3}=\pm \sqrt[4]{\frac{t-1}{t+1}}$

 The points of contact of the given ellipse and hyperbola is $\left(\pm \sqrt[4]{\frac{t+1}{t-1}},\pm \sqrt[4]{\frac{t-1}{t+1}}\right)$

 Substitute this points in the equation of the ellipse $\frac{x^{2}}{t+1}+\frac{y^{2}}{t-1}=1$,

 $\frac{\left(\pm \sqrt[4]{\frac{t+1}{t-1}}\right)^{2}}{t+1}+\frac{\left(\pm \sqrt[4]{\frac{t-1}{t+1}}\right)^{2}}{t-1}=1⟹\frac{1}{\sqrt{t^{2}-1}}+\frac{1}{\sqrt{t^{2}-1}}=1⟹\frac{2}{\sqrt{t^{2}-1}}=1⟹t=\sqrt{5}$, since $t>1$.

**Method 2**

 $xy=1⟹y=\frac{1}{x}$

 Substitute in $\frac{x^{2}}{t+1}+\frac{y^{2}}{t-1}=1$, $\frac{x^{2}}{t+1}+\frac{\left(1/x\right)^{2}}{t-1}=1$

 $\left(t-1\right)\left(x^{2}\right)^{2}-\left(t^{2}-1\right)\left(x^{2}\right)+\left(t+1\right)=0 ….(1)$

 Since $\frac{x^{2}}{t+1}+\frac{y^{2}}{t-1}=1$ is tangential to the hyperbola $xy=1$, we have

 $∆ of (1)=\left(t^{2}-1\right)^{2}-4\left(t-1\right)\left(t+1\right)=0$

$$\left(t^{2}-1\right)^{2}-4\left(t-1\right)\left(t+1\right)=0$$

 $∴ t=\sqrt{5}$ , since $t>1$.

**Method 3**

 The parametric form of $\frac{x^{2}}{t+1}+\frac{y^{2}}{t-1}=1$ is $\left\{\begin{array}{c}x=\sqrt{t+1}\cos(θ)\\y=\sqrt{t-1}\sin(θ)\end{array}\right.$, $0\leq θ<2π$ .

 Substitute in $xy=1$ , we get $\left(\sqrt{t+1}\cos(θ)\right)\left(\sqrt{t-1}\sin(θ)\right)=1⟹\sin(2θ=\frac{2}{\sqrt{t^{2}-1}})$ .

 Since $\frac{x^{2}}{t+1}+\frac{y^{2}}{t-1}=1$ is tangential to the hyperbola $xy=1$, we have

 $\sin(2θ=1⟹)\frac{2}{\sqrt{t^{2}-1}}=1⟹t=\sqrt{5}$ , since $t>1$.

 (For $\sin(2θ=1)$, we have one root. For other values of $\sin(2θ)$ , we can get two roots or no root.)

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