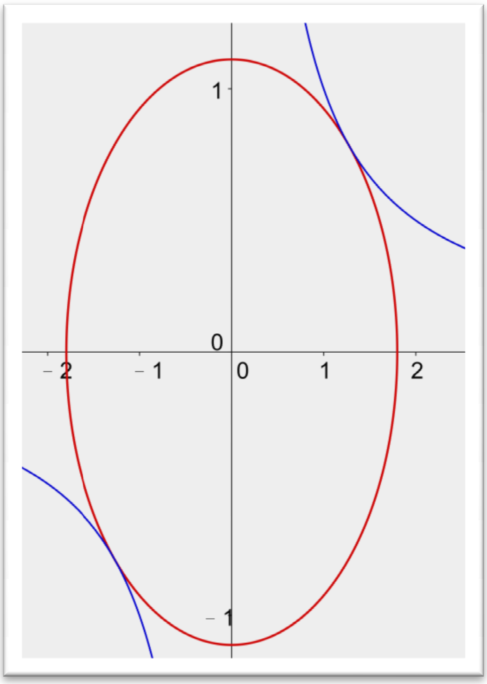
Conics

Given that the ellipse $\frac{x^2}{t+1}+\frac{y^2}{t-1}=1$ is tangential to the hyperbola $xy=1$ and $t>1$.

Find the value of t.



Method 1

$$xy=1 \Longrightarrow y=\frac{1}{x} \Longrightarrow \frac{dy}{dx}=-\frac{1}{x^2}$$

$$\frac{x^2}{t+1}+\frac{y^2}{t-1}=1 \Longrightarrow \frac{2x}{t+1}+\frac{2y}{t-1}\frac{dy}{dx}=0 \Longrightarrow \frac{dy}{dx}=-\frac{x(t-1)}{y(t+1)}$$

Since $\frac{x^2}{t+1}+\frac{y^2}{t-1}=1$ is tangential to the hyperbola $xy=1$, we have

$$-\frac{1}{x^2}=-\frac{x(t-1)}{y(t+1)} \Longrightarrow y=\frac{t-1}{t+1}x^3 \quad \ldots.(1)$$

Substitute (1) in $xy=1$, $\quad \frac{t-1}{t+1}x^4=1 \Longrightarrow x=\pm\sqrt[4]{\frac{t+1}{t-1}}$, since $t>1$

Substitute in (1), $y=\frac{t-1}{t+1}\left(\pm\sqrt[4]{\frac{t+1}{t-1}}\right)^3=\pm\sqrt[4]{\frac{t-1}{t+1}}$

The points of contact of the given ellipse and hyperbola is $\left(\pm\sqrt[4]{\frac{t+1}{t-1}},\pm\sqrt[4]{\frac{t-1}{t+1}}\right)$

Substitute this points in the equation of the ellipse $\frac{x^2}{t+1}+\frac{y^2}{t-1}=1$,

$$\frac{\left(\pm\sqrt[4]{\frac{t+1}{t-1}}\right)^2}{t+1}+\frac{\left(\pm\sqrt[4]{\frac{t-1}{t+1}}\right)^2}{t-1}=1 \Longrightarrow \frac{1}{\sqrt{t^2-1}}+\frac{1}{\sqrt{t^2-1}}=1 \Longrightarrow \frac{2}{\sqrt{t^2-1}}=1 \Longrightarrow t=\sqrt{5}, \text{ since } \quad t>1.$$

Method 2

$$xy = 1 \Longrightarrow y = \frac{1}{x}$$

Substitute in $\frac{x^2}{t+1} + \frac{y^2}{t-1} = 1$, $\frac{x^2}{t+1} + \frac{(1/x)^2}{t-1} = 1$

$$(t-1)(x^2)^2 - (t^2-1)(x^2) + (t+1) = 0 \quad \ldots.(1)$$

Since $\frac{x^2}{t+1} + \frac{y^2}{t-1} = 1$ is tangential to the hyperbola $xy = 1$, we have

$$\Delta \text{ of } (1) = (t^2-1)^2 - 4(t-1)(t+1) = 0$$

$$(t^2-1)^2 - 4(t-1)(t+1) = 0$$

$\therefore$ $t = \sqrt{5}$, since $t > 1$.

Method 3

The parametric form of $\frac{x^2}{t+1} + \frac{y^2}{t-1} = 1$ is $\begin{cases} x = \sqrt{t+1}\cos\theta \\ y = \sqrt{t-1}\sin\theta \end{cases}$, $0 \le \theta < 2\pi$.

Substitute in $xy = 1$, we get $\left(\sqrt{t+1}\cos\theta\right)\left(\sqrt{t-1}\sin\theta\right) = 1 \Longrightarrow \sin 2\theta = \frac{2}{\sqrt{t^2-1}}$.

Since $\frac{x^2}{t+1} + \frac{y^2}{t-1} = 1$ is tangential to the hyperbola $xy = 1$, we have

$$\sin 2\theta = 1 \Longrightarrow \frac{2}{\sqrt{t^2-1}} = 1 \Longrightarrow t = \sqrt{5} \text{ , since } t > 1.$$

(For $\sin 2\theta = 1$, we have one root. For other values of $\sin 2\theta$, we can get two roots or no root.)

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