Limit

Evaluate:

1. $\lim_{n\to\infty}\left[\frac{1}{1\times2}+\frac{1}{2\times3}+\cdots+\frac{1}{n\times(n+1)}\right]$

2. $\lim_{n\to\infty}\left[\frac{1}{2^2-1}+\frac{1}{3^2-1}+\cdots+\frac{1}{n^2-1}\right]$

3. $\lim_{n\to\infty}\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\times\ldots\times\left(1-\frac{1}{n^2}\right)$

4. $\lim_{n\to\infty}\frac{a^n+b^n}{a^{n+1}+b^{n+1}}$

5. $\lim_{n\to\infty}\left[\sqrt[3]{2n+1}-\sqrt[3]{2n-1}\right]$

6. $\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots+\frac{1}{(2n)^2}\right]$

7. $\lim_{n\to\infty}\left[\frac{\sqrt{1}+\sqrt{2}+\cdots\sqrt{n-1}}{n\sqrt{n}}\right]$

8. $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right]$

9. $\lim_{n\to\infty}\left[\frac{1\times3\times5\times\ldots\times(2n-1)}{2\times4\times6\times\ldots\times(2n)}\right]$

1. $\lim_{n\to\infty}\left[\frac{1}{1\times2}+\frac{1}{2\times3}+\cdots+\frac{1}{n\times(n+1)}\right]$

$$=\lim_{n\to\infty}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$=\lim_{n\to\infty}\left[1-\frac{1}{n+1}\right]=1$$

2. $\lim_{n\to\infty}\left[\frac{1}{2^2-1}+\frac{1}{3^2-1}+\cdots+\frac{1}{n^2-1}\right]=\lim_{n\to\infty}\sum_{r=2}^{n}\frac{1}{(r-1)(r+1)}$

$$=\lim_{n\to\infty}\sum_{r=2}^{n}\frac{1}{2}\left[\frac{1}{r-1}-\frac{1}{r+1}\right]=\frac{1}{2}\lim_{n\to\infty}\left[\sum_{r=2}^{n}\frac{1}{r-1}-\sum_{r=2}^{n}\frac{1}{r+1}\right]$$

$$=\frac{1}{2}\lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\sum_{r=3}^{n}\frac{1}{r-1}\right)-\left(\sum_{r=2}^{n-2}\frac{1}{r+1}+\frac{1}{n}+\frac{1}{n+1}\right)\right]$$

$$=\frac{1}{2}\lim_{n\to\infty}\left[\left(1+\frac{1}{2}++\frac{1}{n}+\frac{1}{n+1}\right)-\left(\sum_{r=2}^{n-2}\frac{1}{r+1}+\sum_{r=3}^{n}\frac{1}{r-1}\right)\right]$$

$$=\frac{1}{2}\lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\frac{1}{n}+\frac{1}{n+1}\right)-(0)\right]=\frac{1}{2}\left(1+\frac{1}{2}\right)=\frac{3}{4}$$

3. $\lim_{n\to\infty}\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\times\ldots\times\left(1-\frac{1}{n^2}\right)=\lim_{n\to\infty}\prod_{r=2}^{n}\left(1-\frac{1}{r^2}\right)$

$$= \lim_{n\to\infty} \prod_{r=2}^{n}\left(\frac{r^2-1}{r^2}\right) = \lim_{n\to\infty} \frac{\prod_{r=2}^{n}(r-1)\prod_{r=2}^{n}(r+1)}{\prod_{r=2}^{n} r \prod_{r=2}^{n} r} = \lim_{n\to\infty} \frac{(n-1)!\left[\frac{1}{2}(n+1)!\right]}{n!n!}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2}(n+1)}{n} = \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{1}{2}$$

4. $L = \lim_{n\to\infty} \frac{a^n+b^n}{a^{n+1}+b^{n+1}}$

(a) When $|a| < |b|$,

$$L = \lim_{n\to\infty} \frac{\left(\frac{a}{b}\right)^n+1}{a\left(\frac{a}{b}\right)^n+b} = \frac{1}{b}$$

(b) When $|a| = |b|$,

(i) When a = b,

$$L = \lim_{n\to\infty} \frac{2a^n}{2a^{n+1}} = \lim_{n\to\infty} \frac{1}{a} = \frac{1}{a} \quad \left(\text{or } \frac{1}{b}\right)$$

(ii) When a = -b,

$$L = \lim_{n\to\infty} \frac{a^n+(-a)^n}{a^{n+1}+(-a)^{n+1}} = \frac{1}{a}\lim_{n\to\infty} \frac{1+(-1)^n}{1+(-1)^{n+1}}$$

The sequence oscillates between 0 and $\frac{1}{2}$. The limit does not exist.

(c) When $|a| > |b|$,

$$L = \lim_{n\to\infty} \frac{1+\left(\frac{b}{a}\right)^n}{a+b\left(\frac{b}{a}\right)^n} = \frac{1}{a}$$

5. **Method 1**

$L = \lim_{n\to\infty} \left[\sqrt[3]{2n+1} - \sqrt[3]{2n-1}\right]$

Let $a = \sqrt[3]{2n+1},\ b = \sqrt[3]{2n-1}$

Then $\sqrt[3]{2n+1} - \sqrt[3]{2n-1} = a - b = \frac{a^3-b^3}{a^2+ab+b^2} = \frac{(2n+1)-(2n-1)}{a^2+ab+b^2} = \frac{2}{a^2+ab+b^2}$

As $n \to \infty, a \to \infty \text{ and } b \to \infty$

Hence $L = 0$

Method 2 Put $n = \frac{1}{\Delta x}$

As $n \to \infty, \Delta x \to 0$

$$L = \lim_{\Delta x\to 0}\left[\sqrt[3]{\frac{2}{\Delta x}+1} - \sqrt[3]{\frac{2}{\Delta x}-1}\right] = \lim_{\Delta x\to 0}\left[\sqrt[3]{\frac{2+\Delta x}{\Delta x}} - \sqrt[3]{\frac{2-\Delta x}{\Delta x}}\right]$$

$$= \lim_{\Delta x\to 0}\left[\frac{\sqrt[3]{2+\Delta x}-\sqrt[3]{2}}{\sqrt[3]{\Delta x}} - \frac{\sqrt[3]{2-\Delta x}-\sqrt[3]{2}}{\sqrt[3]{\Delta x}}\right]$$

$$= \lim_{\Delta x\to 0}(\Delta x)^{2/3}\left[\lim_{\Delta x\to 0}\frac{\sqrt[3]{2+\Delta x}-\sqrt[3]{2}}{\Delta x} - \lim_{\Delta x\to 0}\frac{\sqrt[3]{2-\Delta x}-\sqrt[3]{2}}{\Delta x}\right]$$

$$= \lim_{\Delta x\to 0}(\Delta x)^{2/3}\left[\frac{d}{dx}\sqrt[3]{x}\Big|_{x=2} - \frac{d}{dx}\sqrt[3]{x}\Big|_{x=2}\right] = 0$$

6. $L=\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots+\frac{1}{(2n)^2}\right]$

Since $\frac{1}{(2n)^2}\le\frac{1}{(n+k)^2}\le\frac{1}{n^2}$ for all $k=0,1,2,\dots,n$

$\lim_{n\to\infty}\left[\frac{1}{(2n)^2}+\frac{1}{(2n)^2}+\cdots+\frac{1}{(2n)^2}\right]\le L\le\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{n^2}+\cdots+\frac{1}{n^2}\right]$

$\lim_{n\to\infty}\left[\frac{n+1}{(2n)^2}\right]\le L\le\lim_{n\to\infty}\left[\frac{n+1}{n^2}\right]$

$\lim_{n\to\infty}\left[\frac{1+\frac{1}{n}}{4n}\right]\le L\le\lim_{n\to\infty}\left[\frac{1+\frac{1}{n}}{n}\right]$

By Sandwich Theorem, $L=0$

7. $L=\lim_{n\to\infty}\left[\frac{\sqrt{1}+\sqrt{2}+\cdots\sqrt{n-1}}{n\sqrt{n}}\right]=\lim_{n\to\infty}\frac{1}{n}\left[\sqrt{\frac{1}{n}}+\sqrt{\frac{n}{n}}+\cdots\sqrt{\frac{n-1}{n}}\right]=\int_0^1\sqrt{x}dx=\frac{2}{3}$

8. $L=\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right]=\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{\sqrt{1+\left(\frac{1}{n}\right)^2}}+\frac{1}{\sqrt{1+\left(\frac{2}{n}\right)^2}}+\cdots+\frac{1}{\sqrt{1+\left(\frac{n}{n}\right)^2}}\right]$

$=\int_0^1\frac{1}{\sqrt{1+x^2}}dx$

Let $x=\tan u$, $dx=\sec^2 udu$

$L=\int_0^{\pi/4}\frac{\sec^2 udu}{\sec u}=\int_0^{\pi/4}\sec u\,du=\ln|\sec u+\tan u|\Big|_0^{\pi/4}=\ln\left(1+\sqrt{2}\right)$

9. Prove by induction, $0<\frac{1\times3\times5\times\ldots\times(2n-1)}{2\times4\times6\times\ldots\times(2n)}<\frac{1}{\sqrt{2n+1}}$

Let P(n): $\frac{1\times3\times5\times\ldots\times(2n-1)}{2\times4\times6\times\ldots\times(2n)}<\frac{1}{\sqrt{2n+1}}$

For P(1), $\frac{1}{2}<\frac{1}{\sqrt{3}}\sim0.5773502691896$

Assume P(k) is true for some $k\epsilon\mathbf{N}$, that is, $\frac{1\times3\times5\times\ldots\times(2k-1)}{2\times4\times6\times\ldots\times(2k)}<\frac{1}{\sqrt{2k+1}}$ .... (*)

For P(k+1), $\frac{1\times3\times5\times\ldots\times(2k-1)(2k+1)}{2\times4\times6\times\ldots\times(2k)(2k+2)}<\frac{1}{\sqrt{2k+1}}\frac{2k+1}{2k+2}=\frac{\sqrt{2k+1}}{2k+2}<\frac{\sqrt{2k+1}}{\sqrt{2k+1}\sqrt{2k+3}}=\frac{1}{\sqrt{2k+3}}$

since $(2k+2)^2=(2k)^2+4(2k)+4>(2k)^2+4(2k)+3=(2k+1)(2k+3)$

Therefore P(k+1) is true.

By the Principle of Mathematical Induction, P(n) is true for all $n\epsilon\mathbf{N}$

Hence, we have $0\le\lim_{n\to\infty}\left[\frac{1\times3\times5\times\ldots\times(2n-1)}{2\times4\times6\times\ldots\times(2n)}\right]\le\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n+1}}\right]$

By Sandwich Theorem, $\lim_{n\to\infty}\left[\frac{1\times3\times5\times\ldots\times(2n-1)}{2\times4\times6\times\ldots\times(2n)}\right]=0$

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