# Matrix 1 (Easy questions)

**1.** Write out the matrix $(c_{ij})_{3\times 5}$ with $c_{ij} = \begin{cases} 2i - j & \text{if } i = 1 \text{ or } 2 \\ i + 3j & \text{otherwise} \end{cases}$.

**2.** Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -1 \end{pmatrix}$. Find $AA^t$ and $A^tA$.

**3.** Let $M = \begin{pmatrix} 0 & 2 & -3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$. Find $M^n$ for $n = 2, 3, 4, \ldots$

**4.** Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$. Find all matrices $B$ such that $AB =$

**(a)** $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

**(b)** $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

**(c)** $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.

**5.** For any n×n matrices $A$ and $B$, we define a new operation as follows:

$$[A, B] = AB - BA.$$

Let $C$ be another n×n matrix, find

**(a)** $[[A, B], C]$

**(b)** $[[A, B], C] + [[B, C], A] + [[C, A], B]$.

# Matrices : Solution

**1.** In each of the first 2 rows, the entries decrease by 1 when going from any one column to the next column to its right; in the third row, the entries increase by 3 each time instead.

$$\begin{pmatrix} 1 & 0 & -1 & -2 & -3 \\ 3 & 2 & 1 & 0 & -1 \\ 6 & 9 & 12 & 15 & 18 \end{pmatrix}$$

**2.** Note that both $AA^t$ and $A^tA$ are symmetric matrices. (Why?)
So we need only calculate those entries on and above (or below, for that matter) the diagonal and then obtain the other entries by symmetry.

$$AA^t = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -4 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 14 & -11 \\ -11 & 17 \end{pmatrix}.$$

$$A^tA = \begin{pmatrix} 1 & 0 \\ 2 & -4 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 20 & 10 \\ 3 & 10 & 10 \end{pmatrix}.$$

**3.** $M^2 = \begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. $M^3 = O_{3\times3}$, the $3\times3$ zero matrix. For $n \geq 4$, $M^n = M^{n-3}M^3 = M^{n-3}O_{3\times3} = O_{3\times3}$.

**4.** The best way to do this problem is to consider the following question first:
Find x and y such that $\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
We've $\begin{pmatrix} 2x+3y \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Equating corresponding entries on the two sides we get two equations in x and y. We then solve the equations to get $x = 1$ and $y = -1/3$.

**(a)** As the *columns* of B operate independently of one another in the product AB, $A\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ means $A\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $A\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. According to the calculation at the beginning,
$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -1/3 \end{pmatrix}$. So $B = \begin{pmatrix} 1 & 1 \\ -1/3 & -1/3 \end{pmatrix}$.

**(b)** Similarly we get $B = \begin{pmatrix} 1 & 1 & 1 \\ -1/3 & -1/3 & -1/3 \end{pmatrix}$.

**(c)** First note that $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ if and only if $A\begin{pmatrix} -x \\ -y \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ (because $A\begin{pmatrix} -x \\ -y \end{pmatrix} = A\left((-1)\begin{pmatrix} x \\ y \end{pmatrix}\right) = (-1)\,A\begin{pmatrix} x \\ y \end{pmatrix}$ by properties of scalar multiplication and matrix multiplication).
So $B = \begin{pmatrix} 1 & -1 \\ -1/3 & 1/3 \end{pmatrix}$.

**5.** **(a)** $[\,[A, B], C\,]$
$= [\,AB - BA, C\,]$
$= (AB - BA)C - C(AB - BA)$
$= ABC - BAC - CAB + CBA$.

**(b)** We've from (a), $[\,[A, B], C\,] = ABC - BAC - CAB + CBA$.
Similarly, $[\,[B, C], A\,] = BCA - CBA - ABC + ACB$
and $[\,[C, A], B\,] = CAB - ACB - BCA + BAC$.
When we add them up, everything gets cancelled out and the answer is the $n\times n$ zero matrix.

*Note*. The second equality can be obtained from the first (the result of (a)) by changing A to B, B to C and C back to A, i.e. $A \to B \to C \to A$. After that, the third equality is obtained from the second by doing the juggling again. Such a rearrangement is called a cyclic permutation

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