Pure Mathematics

1. Find the determinant in factorized form: $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$.

2. Let $A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 1 & 2 & 3 \end{pmatrix}$. Suppose a, b, c, d are such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$.

(a) Find a, b, c and d.

(b) Hence show that A is non-singular and find its inverse.

[*Question to enjoy at home*: Can you construct A^{-1} without using the values of a, b, c, d and the formula for the inverse?]

3. **(a)** Suppose $A_{2\times2}$ commutes with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Show that A is a diagonal matrix.

(b) Suppose $A_{2\times2}$ commutes with all 2×2 matrices. What can you say about A ?

4. A 2×2 matrix M has the following property:

Whenever $M = PQ$ for some 2×2 matrices P and Q,
we also have $M = QP$.

(a) Show that if X is a non-singular 2×2 matrix, then $MX = XM$.
[Hint. Choose P to be a matrix related to X and then Q to be such that M=PQ.]

(b) **(i)** Let B be any 2×2 matrix. Show that there always exists a scalar x such that $B + xI$ is non-singular, where I denotes the 2×2 identity matrix.
[Hint. Look at the determinant of B+xI.]

(ii) Show that if $A_{2\times2}$ commutes with $B_{2\times2}$ and $C_{2\times2}$, then A also commutes with $hB + kC$ for all scalars h and k .

(iii) Using **(i)**, **(ii)** and **(a)**, show that in fact M commutes with *all* 2×2 matrices.

5. Show that the composite transformation of 2 shears along the x-axis is again a shear along the x-axis.

6. Find the matrix of each of the following transformations:

(a) projection onto the x-axis

(b) projection onto the line $y = x$ (the "diagonal" of the plane).

Matrices : Solution

1. det = (a–b)(b–c)(c–a)(ab+bc+ca).

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2. **(a)** From the given condition we get $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}^{-1} = \ldots = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$. So a = b = 1, c = 3 and d = 2.

(b) detA = $\begin{vmatrix} 1 & 1 & 0 \\ 3 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix} = \ldots = -3 \neq 0$. So A is non-singular. And A^{-1} = (1/detA)(adjA) = ...

$= \begin{pmatrix} -2 & 1 & 0 \\ 3 & -1 & 0 \\ -4/3 & 1/3 & 1/3 \end{pmatrix}$.

Note. A contains $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a "submatrix". Let's guess that correspondingly A^{-1} (if it exists) contains the inverse of that, namely $\begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$, as its own "submatrix". So consider a matrix B of the form

$$\begin{pmatrix} -2 & 1 & 0 \\ 3 & -1 & 0 \\ x & y & z \end{pmatrix}.$$

It's easy to see that AB = $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} -2 & 1 & 0 \\ 3 & -1 & 0 \\ x & y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & * \end{pmatrix}$ by using the given condition. (No need to find the values of a, b, c and d at all.) For the product to equal the identity matrix, we want the third row of the right side to be (0 0 1). This means

(1)(–2) + (2)(3) + 3x = 0, (1)(1) + (2)(–1) + 3y = 0 and 3z = 1,

giving x = –4/3, y = 1/3 and z = 1/3. With these values of x, y and z, we've B the inverse of A.

So $A^{-1} = \begin{pmatrix} -2 & 1 & 0 \\ 3 & -1 & 0 \\ -4/3 & 1/3 & 1/3 \end{pmatrix}$.

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3. **(a)** Write A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, i.e. $\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. This shows b = 0 and c = 0. So A = $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ is a diagonal matrix.

(b) A is given to commute with all 2×2 matrices. So A commutes with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. By (a), A is of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, a diagonal matrix. Now A also commutes with $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$: $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, i.e. $\begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$. So a = d. Hence A is of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, i.e. A = a I, where I is the 2×2 identity matrix and a is any scalar (and such a matrix clearly commutes with all 2×2 matrices).

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4. **(a)** Let X be a non-singular 2×2 matrix. We've to show that MX = XM.
To this end we take P = X^{-1} and Q = XM. We then have PQ = $X^{-1}XM$ = I M = M. So by the given property of M, we also have M = QP = XMX^{-1}, which implies MX = $XMX^{-1}X$ = XMI = XM.

(b) **(i)** It's easy to see that det(B+xI) is a quadratic expression in x when expanded. As the quadratic equation det(B+xI) = 0 has at most 2 roots, there must be some (in fact infinitely many) values of x for which det(B+xI) $\neq 0$. For such x, B+xI is non-singular.

(ii) Suppose A commutes with B and C. Then AB = BA and AC = CA. It follows that A(hB+kC) = A(hB) + A(kC) = h(AB) + k(AC) = h(BA) + k(CA) = (hB)A + (kC)A = (hB+kC)A.
So A commutes with hB+kC.

(iii) Let X be any 2×2 matrix. By (i), there is some scalar x such that X+xI is non-singular. Then M commutes with X+xI by (a). Now M clearly commutes with I too. By (ii), M then commutes with (1)(X+xI) + (-x)I, i.e. with X.
Alternatively, once we know M commutes with X+xI, we've M(X+xI) = (X+xI)M, and hence MX + xM = XM + xM. Cancelling xM, we get MX = XM, i.e. M commutes X as well. (This way you don't even need (ii).)

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5. Let two shears along the x-axis be given by the matrices $\begin{pmatrix}1 & a\\0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & b\\0 & 1\end{pmatrix}$. Then the matrix of their composite is
$\begin{pmatrix}1 & b\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & a+b\\0 & 1\end{pmatrix}$, which shows that the composite is again a shear along the x-axis.

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6. **(a)** The projection onto the x-axis takes $\begin{pmatrix}x\\y\end{pmatrix}$ to $\begin{pmatrix}x'\\y'\end{pmatrix}$, where x' = x and y = 0. So the matrix of the transformation is $\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$.

(b) We can regard the projection onto the line y = x as the composite of three transformations: first the rotation about the origin through –45°, then the projection onto the x-axis and finally the rotation about the origin through +45°. So the required matrix is equal to

$$\begin{pmatrix}\cos 45^\circ & -\sin 45^\circ\\ \sin 45^\circ & \cos 45^\circ\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}\cos(-45^\circ) & -\sin(-45^\circ)\\ \sin(-45^\circ) & \cos(-45^\circ)\end{pmatrix}$$

$$= \begin{pmatrix}1/2 & 1/2\\1/2 & 1/2\end{pmatrix}.$$

Alternatively, the image point $\begin{pmatrix}x'\\y'\end{pmatrix}$ of a point P$\begin{pmatrix}x\\y\end{pmatrix}$ under this projection is just the midpoint of P and its image under the reflection in y = x, which is the point $\begin{pmatrix}y\\x\end{pmatrix}$. So
$\begin{pmatrix}x'\\y'\end{pmatrix} = [\ \begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}y\\x\end{pmatrix}] / 2 = \begin{pmatrix}x/2+y/2\\x/2+y/2\end{pmatrix} = \begin{pmatrix}1/2 & 1/2\\1/2 & 1/2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$.
The required matrix is therefore $\begin{pmatrix}1/2 & 1/2\\1/2 & 1/2\end{pmatrix}$.