**Polygons**

**1.** (Warm-up on Pythagoras Theorem)

 Given a regular octagon ABCDEFGH with

 sides 2 cm.

 **(a)** If AE intersects CG at X.

 Find the length of AX.

 **(b)** If AD cuts CG at Y.

Find the length of XY.



**1. (a)** Draw a square outside touching the sides of the octagon as in the diagram.

 It can be seen that X is the centre of both

 the octagon and the square.

 By Pythagoras Theorem on $∆PAH$,

 $PA=PH=\sqrt{2}$

 $PQ=PA+AB+BQ=\sqrt{2}+2+\sqrt{2}$

 $=2+2\sqrt{2}$

 So the side of the square is $2+2\sqrt{2}$.

 By Pythagoras Theorem on $∆ABE$,

 $AE=\sqrt{2^{2}+2+\left(2\sqrt{2}\right)^{2}}=\sqrt{8 \sqrt{2}+16}$

 $AX=\frac{1}{2}AE=\frac{1}{2}\sqrt{8 \sqrt{2}+16}=\sqrt{2 \sqrt{2}+4}≈2.6131259297528$ **cm**

 **(b)** Note that $∆ABY$ is right angled and $∠BAY=45°$.

 By Pythagoras Theorem, $AY=\sqrt{8}$

 Note that AE is perpendicular to CG.

 Apply Pythagoras Theorem on $∆AXY$ .

 $XY=\sqrt{\left(\sqrt{8}\right)^{2}-\left(\sqrt{2 \sqrt{2}+4}\right)^{2}}=\sqrt{4-2 \sqrt{2}}≈1.0823922002924$ **cm**

**2.** A dodecagon of is placed inside a circle of radius 1 cm, and the twelve dividing points are joined to the circle's centre, producing twelve rays. Starting from $P\_{1}$ a segment is drawn perpendicular to the next ray $OP\_{2}$ in the anti-clockwise sense; and from the foot of this perpendicular another perpendicular segment is drawn to the next ray, and so on forever.

** Taking** $Q\_{1}=P\_{1}$.

 **(a)** Find the limit of the sum of the lengths of these segments:

 $Q\_{1}Q\_{2}+Q\_{2}Q\_{3}+Q\_{3}Q\_{4}+Q\_{4}Q\_{5}+…$

 $=\sum\_{k=1}^{\infty }Q\_{k}Q\_{k+1}$

 **(b)** Find the limit of the area of the triangles :

 $∆OQ\_{1}Q\_{2}+∆OQ\_{2}Q\_{3}+∆OQ\_{3}Q\_{4}+…$

 $=\sum\_{k=1}^{\infty }∆OQ\_{k}Q\_{k+1}$.

 **(c)** (For more able students)

 Instead of starting with the circle divided into twelve equal parts, we now to divide it into n equal parts. Let $∠Q\_{1}OQ\_{2}=α$.

 **(i)** Find the sum of the lengths: $ \sum\_{k=1}^{\infty }Q\_{k}Q\_{k+1}$

 **(ii)** Find the limit of the area of the triangles : $\sum\_{k=1}^{\infty }∆OQ\_{k}Q\_{k+1}$

**2. (a)** Consider the triangle $∆OQ\_{k}Q\_{k+1}$ .

 $∠OQ\_{k}Q\_{k+1}=60°,∠Q\_{k}OQ\_{k+1}=30°,∠Q\_{k}Q\_{k+1}O=90°$

 Let $OQ\_{k}=r\_{k}, OQ\_{k+1}=r\_{k+1}$.

 Then $r\_{k+1}=r\_{k}\cos(30°)=\frac{\sqrt{3}}{2}r\_{k}$

 $Q\_{k}Q\_{k+1}=r\_{k}\sin(30°)=\frac{1}{2}r\_{k}$

 $Q\_{k+1}Q\_{k+2}=\frac{1}{2}r\_{k+1}=\frac{1}{2}\left(\frac{\sqrt{3}}{2}r\_{k}\right)=Q\_{k}Q\_{k+1}\frac{\sqrt{3}}{2}$

 Sum of the lengths:

 $ \sum\_{k=1}^{\infty }Q\_{k}Q\_{k+1}=Q\_{1}Q\_{2}+Q\_{2}Q\_{3}+Q\_{3}Q\_{4}+Q\_{4}Q\_{5}+…$

 $=\frac{1}{2}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^{2}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^{3}+…$ , which is an infinite geometric series

 $=\frac{1}{2}\left(\frac{1}{1-\frac{\sqrt{3}}{2}}\right)$ , where the common ratio $\frac{\sqrt{3}}{2}<1$.

 $=\frac{1}{2-\sqrt{3}}=2+\sqrt{3}≈3.7320508075689$ cm

 **(b)** $OQ\_{k+1}=r\_{k+1}=\frac{\sqrt{3}}{2}r\_{k}$

 The limit of the area of the triangles : $∆OQ\_{1}Q\_{2}+∆OQ\_{2}Q\_{3}+∆OQ\_{3}Q\_{4}+…$

 $=\frac{1}{2}OQ\_{2}×Q\_{1}Q\_{2}+\frac{1}{2}OQ\_{3}×Q\_{2}Q\_{3}+\frac{1}{2}OQ\_{4}×Q\_{3}Q\_{4}+…$

 $=\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^{2}\left[\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right]+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^{3}\left[\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^{2}\right]+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^{4}\left[\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^{3}\right]+…$

 $=\left(\frac{1}{2}\right)^{2}\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{2}\right)^{2}\left(\frac{\sqrt{3}}{2}\right)^{3}+\left(\frac{1}{2}\right)^{2}\left(\frac{\sqrt{3}}{2}\right)^{5}+\left(\frac{1}{2}\right)^{2}\left(\frac{\sqrt{3}}{2}\right)^{7}+…$

 $=\left(\frac{1}{2}\right)^{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{1-\left(\frac{\sqrt{3}}{2}\right)^{2}}\right)$, where the common ratio $\left(\frac{\sqrt{3}}{2}\right)^{2}<1$

 $=\frac{\sqrt{3}}{2}≈0.8660254037844$ cm

 **(c) (i)** Consider the triangle $∆OQ\_{k}Q\_{k+1}$ .

 $∠Q\_{k}OQ\_{k+1}=α, ∠Q\_{k}Q\_{k+1}O=90°$

 Let $OQ\_{k}=r\_{k}, OQ\_{k+1}=r\_{k+1}$.

 Then $r\_{k+1}=r\_{k}\cos(α)$

 $Q\_{k}Q\_{k+1}=r\_{k}\sin(α)$

 $Q\_{k+1}Q\_{k+2}=r\_{k+1}\sin(α)=r\_{k}\sin(α)\cos(α)=Q\_{k}Q\_{k+1}\cos(α)$

Sum of the lengths:

 $ \sum\_{k=1}^{\infty }Q\_{k}Q\_{k+1}=Q\_{1}Q\_{2}+Q\_{2}Q\_{3}+Q\_{3}Q\_{4}+Q\_{4}Q\_{5}+…$

$=\left(\sin(α)\right)+\left(\sin(α)\right)\left(\cos(α)\right)+\left(\sin(α)\right)\left(\cos(α)\right)^{2}+\left(\sin(α)\right)\left(\cos(α)\right)^{3}+…$

$=\left(\sin(α)\right)\frac{1}{1-\cos(α)}=\frac{\sin(α)}{1-\cos(α)}$

 **(ii)** Note that :$OQ\_{k+1}=r\_{k+1}=r\_{k}\cos(α)$

 The limit of the area of the triangles :

 $∆OQ\_{1}Q\_{2}+∆OQ\_{2}Q\_{3}+∆OQ\_{3}Q\_{4}+…$

 $=\frac{1}{2}OQ\_{2}×Q\_{1}Q\_{2}+\frac{1}{2}OQ\_{3}×Q\_{2}Q\_{3}+\frac{1}{2}OQ\_{4}×Q\_{3}Q\_{4}+…$

 $=\frac{1}{2}\cos(α)\sin(α)+\frac{1}{2}\left(\cos(α)\right)^{2}\left[\sin(α)\cos(α)\right]+\frac{1}{2}\left(\cos(α)\right)^{3}\left[\sin(α)\left(\cos(α)\right)^{2}\right]$

$$+\frac{1}{2}\left(\cos(α)\right)^{4}\left[\sin(α)\left(\cos(α)\right)^{3}\right]+…$$

$=\frac{1}{2}\sin(α)\cos(α)+\frac{1}{2}\sin(α)\left(\cos(α)\right)^{3}+\frac{1}{2}\sin(α)\left(\cos(α)\right)^{5}+\frac{1}{2}\sin(α)\left(\cos(α)\right)^{7}+…$

 $=\frac{1}{2}\sin(α)\cos(α)\frac{1}{1-\left(\cos(α)\right)^{2}}=\frac{\sin(α)\cos(α)}{2\left(\sin(α)\right)^{2}}=\frac{\cos(α)}{2\sin(α)}=\frac{1}{2\tan(α)}$

**Yue Kwok Choy**

**7/3/2017**