**Vector**

**(a)** Let $P$ be a point inside $∆ABC$ such that $3\rightharpoonaccent{PA}+2\rightharpoonaccent{PB}+\rightharpoonaccent{PC}=\rightharpoonaccent{0}$ ,

find $\left[∆PBC\right]:\left[∆PCA\right]:\left[∆PAB\right]$, where $\left[∆PAB\right]$ represents the area of the $∆PAB$.



**(b)** Let $P$ be a point inside $∆ABC$ such that $a\rightharpoonaccent{PA}+b\rightharpoonaccent{PB}+c\rightharpoonaccent{PC}=\rightharpoonaccent{0}$ ,

 find $\left[∆PBC\right]:\left[∆PCA\right]:\left[∆PAB\right]$.

**(c)** If $P$ be a point inside $∆ABC$ such that $\rightharpoonaccent{PA}+5\rightharpoonaccent{PB}+3\rightharpoonaccent{PC}=\rightharpoonaccent{0}$, find

 $\left[∆ABC\right]:\left[ABPCA\right]$, where $ABPCA$ is the concave quadrilateral.

**(a)**

**Method 1**

 $3\rightharpoonaccent{PA}+2\rightharpoonaccent{PB}+\rightharpoonaccent{PC}=\rightharpoonaccent{0}$

 $3\left(-\rightharpoonaccent{AP}\right)+2\left(\rightharpoonaccent{AB}-\rightharpoonaccent{AP}\right)+\left(\rightharpoonaccent{AC}-\rightharpoonaccent{AP}\right)=\rightharpoonaccent{0}$

 $\rightharpoonaccent{AP}=\frac{1}{6}\left(2\rightharpoonaccent{AB}+\rightharpoonaccent{AC}\right)$

Produce $AP$ to meet $BC$ at $D$.

Let $\rightharpoonaccent{AD}=λ\rightharpoonaccent{AP}=\frac{λ}{3}\rightharpoonaccent{AB}+\frac{λ}{6}\rightharpoonaccent{AC}$

Since $B,D,C$ are collinear, $\frac{λ}{3}+\frac{λ}{6}=1$ ,

 $∴λ=2$

 and $\rightharpoonaccent{AD}=\frac{2}{3}\rightharpoonaccent{AB}+\frac{1}{3}\rightharpoonaccent{AC}⟹BD:DC=1:2$

Also, $\rightharpoonaccent{AD}=2\rightharpoonaccent{AP}⟹AD=2AP⟹AP:PD=1:1$

 $\left[∆PAB\right]=k$

 $\left[∆PDB\right]=k$

 $\left[∆ABD\right]=\left[∆PAB\right]+\left[∆PDB\right]=k+k=2k$

 Since $BD:DC=1:2,$ $\left[∆PDC\right]=2k$ and $\left[∆ADC\right]=4k$

 $\left[∆PBC\right]=\left[∆PDB\right]+\left[∆PDC\right]=k+2k=3k$

 $\left[∆PCA\right]=\left[∆ADC\right]-\left[∆PDC\right]=4k-2k=2k$

$$∴ \left[∆PBC\right]:\left[∆PCA\right]:\left[∆PAB\right]=3k:2k:k=3:2:1$$

**Method 2**



Place a coordinate system with $A$ as origin
and $\rightharpoonaccent{AB}$ along x-axis.

Since $3\rightharpoonaccent{PA}+2\rightharpoonaccent{PB}+\rightharpoonaccent{PC}=\rightharpoonaccent{0}$

Consider the y-cordinates,

 $-3y\_{P}-2y\_{P}+\left(y\_{c}-y\_{P}\right)=0$

 $y\_{C}=6y\_{P}$

Hence, $\left[∆PAB\right]=\frac{1}{6}\left[∆ABC\right]$

If we place a coordinate system with $B$ as origin and $\rightharpoonaccent{BC}$ along x-axis, we get: $\left[∆PBC\right]=\frac{1}{2}\left[∆ABC\right]$

If we place a coordinate system with $C$ as origin and $\rightharpoonaccent{CA}$ along x-axis, we get: $\left[∆PCA\right]=\frac{1}{3}\left[∆ABC\right]$

$$∴ \left[∆PBC\right]:\left[∆PCA\right]:\left[∆PAB\right]=3:2:1$$

**Method 3**

Let $\rightharpoonaccent{PA^{'}}=3\rightharpoonaccent{PA}, \rightharpoonaccent{PB^{'}}=2\rightharpoonaccent{PB} , \rightharpoonaccent{PC'}=\rightharpoonaccent{PC } $

Then $3\rightharpoonaccent{PA}+2\rightharpoonaccent{PB}+\rightharpoonaccent{PC}=\rightharpoonaccent{0}⟹ \rightharpoonaccent{PA^{'}}+\rightharpoonaccent{PB^{'}}+\rightharpoonaccent{PC^{'}}=0$

Hence, $P$ is the **centroid** of the $∆A'B'C'$.

 $\left[∆PB'C'\right]:\left[∆PC'A'\right]:\left[∆PA'B'\right]=1:1:1⟹\left[∆PB'C'\right]=\left[∆PC'A'\right]=\left[∆PA'B'\right]=k$

 $\frac{\left[∆PBC\right]}{\left[∆PB'C'\right]}=\frac{\frac{1}{2}PB×PC×sin ∠BPC}{\frac{1}{2}PB'×PC'×sin ∠BP'C'}=\frac{\frac{1}{2}PB×PC×sin ∠BPC}{\frac{1}{2}\left(2PB\right)×PC×sin ∠BPC}=\frac{1}{2}⟹\left[∆PB'C'\right]=\frac{1}{2}k$

Similarly, $\frac{\left[∆PCA\right]}{\left[∆PC'A'\right]}=\frac{1}{3} and \frac{\left[∆PAB\right]}{\left[∆PA'B'\right]}=\frac{1}{6}⟹\left[∆PCA\right]=\frac{1}{3}k and \left[∆PA'B'\right]=\frac{1}{6}k$

 $∴ \left[∆PBC\right]:\left[∆PCA\right]:\left[∆PAB\right]=\frac{1}{2}k:\frac{1}{3}k:\frac{1}{6}k=3:2:1$

**Method 4**



$$3\rightharpoonaccent{PA}+2\rightharpoonaccent{PB}+\rightharpoonaccent{PC}=\rightharpoonaccent{0}⟹\rightharpoonaccent{PB}+\rightharpoonaccent{PC}+2\left(\rightharpoonaccent{PA}+\rightharpoonaccent{PB}\right)=0$$

Construct the mid-point of $BC$ as $D$.

Then $\rightharpoonaccent{PB}+\rightharpoonaccent{PC}=2\rightharpoonaccent{PD}$

Construct the mid-point of $AB$ as $E$.

Then $\rightharpoonaccent{PA}+\rightharpoonaccent{PB}=2\rightharpoonaccent{PE}$

Hence we can get $2\rightharpoonaccent{PD}+4\rightharpoonaccent{PE}=0$

$$∴ \rightharpoonaccent{PD}=2\rightharpoonaccent{EP}$$

Then $D,P,E$ is a straight line and $PD:EP=2:1$

Let $\left[∆PEB\right]=k$, then $\left[∆PDB\right]=2k$

Since $D,E$ are mid-points of $ BC$ and $AB$ respectively,

 $\left[∆APE\right]=k,\left[∆CPD\right]=2k$

Hence $\left[∆PAB\right]=2k, \left[∆PBC\right]=4k$

By Mid-point Theorem, $AC=2ED$ and note that $∆EBD\~∆EDB$

Since $\left[∆EDB\right]=3k,\left[∆ABC\right]=12k$,

$$\left[∆PCA\right]=12k-4k-2k=6k$$

 $∴ \left[∆PBC\right]:\left[∆PCA\right]:\left[∆PAB\right]=6k:4k:2k=3:2:1$

**Method 5** Note: $\left|\begin{matrix}x\_{1}&x\_{2}\\y\_{1}&y\_{2}\end{matrix}\right|=x\_{1}y\_{2}-x\_{2}y\_{1}$

 Let $P$ be the origin and $\rightharpoonaccent{PA}=x\_{1}i+y\_{1}j$ **,** $\rightharpoonaccent{PB}=x\_{2}i+y\_{2}j$ **,** $\rightharpoonaccent{PC}=x\_{3}i+y\_{3}j$

$$3\rightharpoonaccent{PA}+2\rightharpoonaccent{PB}+\rightharpoonaccent{PC}=\rightharpoonaccent{0}⟹\left(3x\_{1}+2x\_{2}+x\_{3}\right)i+\left(3y\_{1}+2y\_{2}+y\_{3}\right)j=0$$

$$⟹\left\{\begin{array}{c}3x\_{1}+2x\_{2}+x\_{3}=0\\3y\_{1}+2y\_{2}+y\_{3}=0\end{array}\right.⟹\left|\begin{matrix}x\_{2}&x\_{3}\\y\_{2}&y\_{3}\end{matrix}\right|:\left|\begin{matrix}x\_{3}&x\_{1}\\y\_{3}&y\_{1}\end{matrix}\right|:\left|\begin{matrix}x\_{1}&x\_{2}\\y\_{1}&y\_{2}\end{matrix}\right|=3:2:1$$

$$∴ \left[∆PBC\right]:\left[∆PCA\right]:\left[∆PAB\right]=\frac{1}{2}\left|\begin{matrix}x\_{2}&x\_{3}\\y\_{2}&y\_{3}\end{matrix}\right|:\frac{1}{2}\left|\begin{matrix}x\_{3}&x\_{1}\\y\_{3}&y\_{1}\end{matrix}\right|:\frac{1}{2}\left|\begin{matrix}x\_{1}&x\_{2}\\y\_{1}&y\_{2}\end{matrix}\right|=3:2:1$$

**Method 6**

There is a one-to-one correspondence between vector and complex number.

For any vector $\rightharpoonaccent{v}=xi+yj$, there is a complex number $z=x+yi$.

The problem then becomes:

Take $P$ to be the origin of the complex plane and $A\left(z\_{1}\right), B\left(z\_{2}\right) and C\left(z\_{3}\right)$, and

$3z\_{1}+2z\_{2}+z\_{3}=0$ , find $\left[∆PBC\right]:\left[∆PCA\right]:\left[∆PAB\right]$, where $\left[∆PAB\right]$ represents the area of the $∆PAB$.

Since $3z\_{1}+2z\_{2}+z\_{3}=0$, we have the conjugate equation $3\overbar{z\_{1}}+2\overbar{z\_{2}}+\overbar{z\_{3}}=0$.

Hence $\left(z\_{2}\overbar{z\_{3}}-z\_{3}\overbar{z\_{2}}\right):\left(z\_{3}\overbar{z\_{1}}-z\_{1}\overbar{z\_{3}}\right):\left(z\_{1}\overbar{z\_{2}}-z\_{2}\overbar{z\_{1}}\right)=3:2:1$.

By the area formula in complex number, we have

 $\left[∆PBC\right]:\left[∆PCA\right]:\left[∆PAB\right]=\frac{i}{4}\left|\begin{matrix}1&1&1\\0&z\_{2}&z\_{3}\\0&\overbar{z\_{2}}&\overbar{z\_{3}}\end{matrix}\right|:\frac{i}{4}\left|\begin{matrix}1&1&1\\0&z\_{3}&z\_{1}\\0&\overbar{z\_{3}}&\overbar{z\_{1}}\end{matrix}\right|:\frac{i}{4}\left|\begin{matrix}1&1&1\\0&z\_{1}&z\_{2}\\0&\overbar{z\_{1}}&\overbar{z\_{2}}\end{matrix}\right|$

 $=\left(z\_{2}\overbar{z\_{3}}-z\_{3}\overbar{z\_{2}}\right):\left(z\_{3}\overbar{z\_{1}}-z\_{1}\overbar{z\_{3}}\right):\left(z\_{1}\overbar{z\_{2}}-z\_{2}\overbar{z\_{1}}\right)$

 $=3:2:1$

**(b)** Place a coordinate system with $A$ as origin and $\rightharpoonaccent{AB}$

 along x-axis.

 Since $a\rightharpoonaccent{PA}+b\rightharpoonaccent{PB}+c\rightharpoonaccent{PC}=\rightharpoonaccent{0}$

 Consider the y-cordinates,

 $-ay\_{P}-by\_{P}+c\left(y\_{c}-y\_{P}\right)=0$

 $y\_{C}=\frac{c}{a+b+c}y\_{P}$

Hence, $\left[∆PAB\right]=\frac{c}{a+b+c}\left[∆ABC\right]$

If we set a coordinate system with $B$ as origin and $\rightharpoonaccent{BC}$ along x-axis, then: $\left[∆PBC\right]=\frac{a}{a+b+c}\left[∆ABC\right]$

If we set a coordinate system with $C$ as origin and $\rightharpoonaccent{CA}$ along x-axis, then: $\left[∆PCA\right]=\frac{b}{a+b+c}\left[∆ABC\right]$

$$∴ \left[∆PBC\right]:\left[∆PCA\right]:\left[∆PAB\right]=\frac{a}{a+b+c}:\frac{b}{a+b+c}:\frac{c}{a+b+c}=a:b:c$$



**(c)** $\rightharpoonaccent{PA}+5\rightharpoonaccent{PB}+3\rightharpoonaccent{PC}=\rightharpoonaccent{0}$, by (**b)**,

$$ \left[∆PBC\right]:\left[∆PCA\right]:\left[∆PAB\right]=1:5:3$$

 $\left[∆PBC\right]=k, \left[∆PCA\right]=5k, \left[∆PAB\right]=3k$

 Therefore, $\left[∆ABC\right]=k+5k+3k=9k$

 and $\left[ABPCA\right]=3k+5k=8k$

 $\left[∆ABC\right]:\left[ABPCA\right]=9k:8k=9:8$

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**Method 6 of part (a) added on 10-3-16**