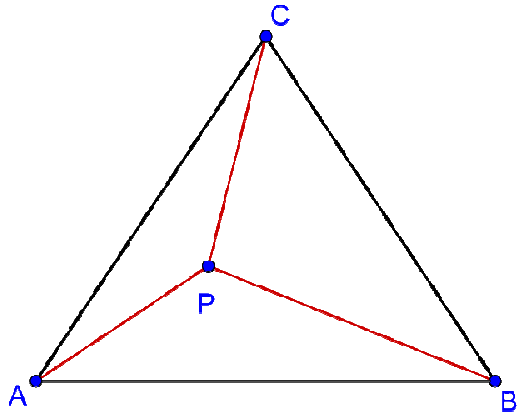
Vector

(a) Let P be a point inside $\Delta\mathrm{ABC}$ such that $3\overrightarrow{\mathrm{PA}}+2\overrightarrow{\mathrm{PB}}+\overrightarrow{\mathrm{PC}}=\vec{0}$,
find $[\Delta\mathrm{PBC}]:[\Delta\mathrm{PCA}]:[\Delta\mathrm{PAB}]$, where $[\Delta\mathrm{PAB}]$ represents the area of the $\Delta\mathrm{PAB}$.



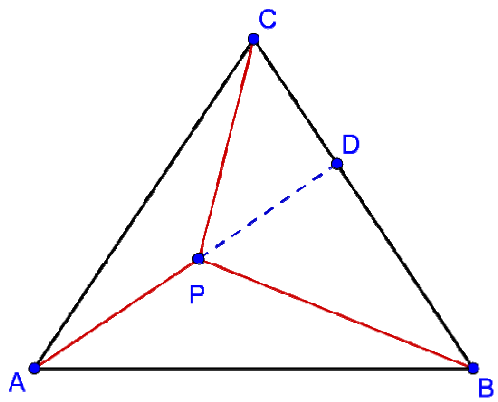
(b) Let P be a point inside $\Delta\mathrm{ABC}$ such that $\mathrm{a}\overrightarrow{\mathrm{PA}}+\mathrm{b}\overrightarrow{\mathrm{PB}}+\mathrm{c}\overrightarrow{\mathrm{PC}}=\vec{0}$,
find $[\Delta\mathrm{PBC}]:[\Delta\mathrm{PCA}]:[\Delta\mathrm{PAB}]$.

(c) If P be a point inside $\Delta\mathrm{ABC}$ such that $\overrightarrow{\mathrm{PA}}+5\overrightarrow{\mathrm{PB}}+3\overrightarrow{\mathrm{PC}}=\vec{0}$, find
$[\Delta\mathrm{ABC}]:[\mathrm{ABPCA}]$, where ABPCA is the concave quadrilateral.

(a)

Method 1

$$3\overrightarrow{PA}+2\overrightarrow{PB}+\overrightarrow{PC}=\vec{0}$$
$$3(-\overrightarrow{AP})+2(\overrightarrow{AB}-\overrightarrow{AP})+(\overrightarrow{AC}-\overrightarrow{AP})=\vec{0}$$
$$\overrightarrow{AP}=\frac{1}{6}(2\overrightarrow{AB}+\overrightarrow{AC})$$



Produce AP to meet BC at D.

Let $\overrightarrow{AD}=\lambda\overrightarrow{AP}=\frac{\lambda}{3}\overrightarrow{AB}+\frac{\lambda}{6}\overrightarrow{AC}$

Since B, D, C are collinear, $\frac{\lambda}{3}+\frac{\lambda}{6}=1$,

$\therefore \lambda=2$

and $\overrightarrow{AD}=\frac{2}{3}\overrightarrow{AB}+\frac{1}{3}\overrightarrow{AC}\Longrightarrow BD:DC=1:2$

Also, $\overrightarrow{AD}=2\overrightarrow{AP}\Longrightarrow AD=2AP\Longrightarrow AP:PD=1:1$

$[\Delta PAB]=k$

$[\Delta PDB]=k$

$[\Delta ABD]=[\Delta PAB]+[\Delta PDB]=k+k=2k$

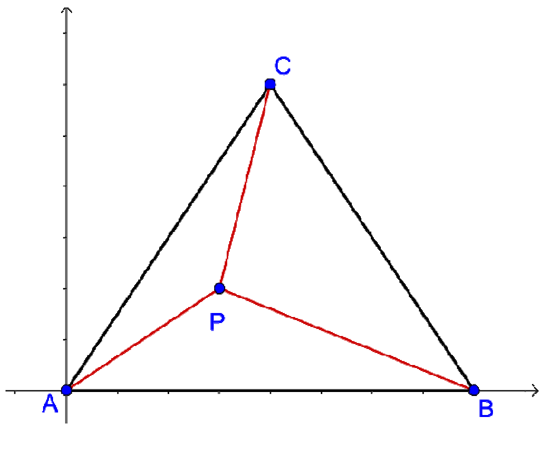
Since $BD:DC=1:2$, $[\Delta PDC]=2k$ and $[\Delta ADC]=4k$

$[\Delta PBC]=[\Delta PDB]+[\Delta PDC]=k+2k=3k$

$[\Delta PCA]=[\Delta ADC]-[\Delta PDC]=4k-2k=2k$

$$\therefore\quad [\Delta PBC]:[\Delta PCA]:[\Delta PAB]=3k:2k:k=3:2:1$$

Method 2



Place a coordinate system with A as origin and $\overrightarrow{AB}$ along x-axis.

Since $3\overrightarrow{PA}+2\overrightarrow{PB}+\overrightarrow{PC}=\vec{0}$

Consider the y-cordinates,

$$-3y_P-2y_P+(y_c-y_P)=0$$
$$y_C=6y_P$$

Hence, $[\Delta PAB]=\frac{1}{6}[\Delta ABC]$

If we place a coordinate system with B as origin and $\overrightarrow{BC}$ along x-axis, we get: $[\Delta PBC]=\frac{1}{2}[\Delta ABC]$

If we place a coordinate system with C as origin and $\overrightarrow{CA}$ along x-axis, we get: $[\Delta PCA]=\frac{1}{3}[\Delta ABC]$

$$\therefore\quad [\Delta PBC]:[\Delta PCA]:[\Delta PAB]=3:2:1$$

Method 3

Let $\overrightarrow{PA'} = 3\overrightarrow{PA}$, $\overrightarrow{PB'} = 2\overrightarrow{PB}$, $\overrightarrow{PC'} = \overrightarrow{PC}$

Then $3\overrightarrow{PA} + 2\overrightarrow{PB} + \overrightarrow{PC} = \vec{0} \Longrightarrow \overrightarrow{PA'} + \overrightarrow{PB'} + \overrightarrow{PC'} = 0$

Hence, P is the **centroid** of the $\Delta A'B'C'$.

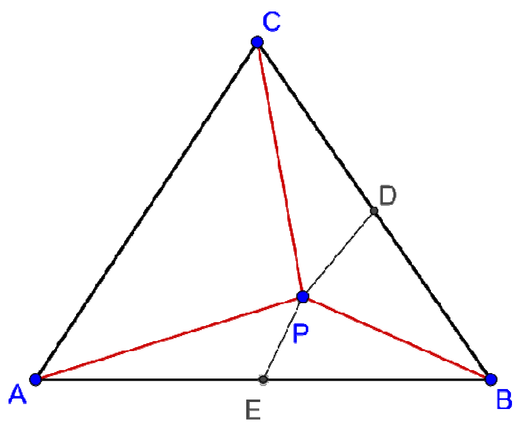
$[\Delta PB'C'] : [\Delta PC'A'] : [\Delta PA'B'] = 1:1:1 \Longrightarrow [\Delta PB'C'] = [\Delta PC'A'] = [\Delta PA'B'] = k$

$$\frac{[\Delta PBC]}{[\Delta PB'C']} = \frac{\frac{1}{2}PB \times PC \times \sin\angle BPC}{\frac{1}{2}PB' \times PC' \times \sin\angle BP'C'} = \frac{\frac{1}{2}PB \times PC \times \sin\angle BPC}{\frac{1}{2}(2PB) \times PC \times \sin\angle BPC} = \frac{1}{2} \Longrightarrow [\Delta PB'C'] = \frac{1}{2}k$$

Similarly, $\frac{[\Delta PCA]}{[\Delta PC'A']} = \frac{1}{3}$ and $\frac{[\Delta PAB]}{[\Delta PA'B']} = \frac{1}{6} \Longrightarrow [\Delta PCA] = \frac{1}{3}k$ and $[\Delta PA'B'] = \frac{1}{6}k$

$\therefore$ $[\Delta PBC] : [\Delta PCA] : [\Delta PAB] = \frac{1}{2}k : \frac{1}{3}k : \frac{1}{6}k = 3:2:1$

Method 4



$3\overrightarrow{PA} + 2\overrightarrow{PB} + \overrightarrow{PC} = \vec{0} \Longrightarrow \overrightarrow{PB} + \overrightarrow{PC} + 2\left(\overrightarrow{PA} + \overrightarrow{PB}\right) = 0$

Construct the mid-point of BC as D.

Then $\overrightarrow{PB} + \overrightarrow{PC} = 2\overrightarrow{PD}$

Construct the mid-point of AB as E.

Then $\overrightarrow{PA} + \overrightarrow{PB} = 2\overrightarrow{PE}$

Hence we can get $2\overrightarrow{PD} + 4\overrightarrow{PE} = 0$

$\therefore$ $\overrightarrow{PD} = 2\overrightarrow{EP}$

Then D, P, E is a straight line and $PD : EP = 2:1$

Let $[\Delta PEB] = k$, then $[\Delta PDB] = 2k$

Since D, E are mid-points of BC and AB respectively,

$[\Delta APE] = k, [\Delta CPD] = 2k$

Hence $[\Delta PAB] = 2k, [\Delta PBC] = 4k$

By Mid-point Theorem, $AC = 2ED$ and note that $\Delta EBD \sim \Delta EDB$

Since $[\Delta EDB] = 3k, [\Delta ABC] = 12k,$

$[\Delta PCA] = 12k - 4k - 2k = 6k$

$\therefore$ $[\Delta PBC] : [\Delta PCA] : [\Delta PAB] = 6k : 4k : 2k = 3:2:1$

Method 5 Note: $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1$

Let P be the origin and $\overrightarrow{PA} = x_1\mathbf{i} + y_1\mathbf{j}$, $\overrightarrow{PB} = x_2\mathbf{i} + y_2\mathbf{j}$, $\overrightarrow{PC} = x_3\mathbf{i} + y_3\mathbf{j}$

$$3\overrightarrow{PA} + 2\overrightarrow{PB} + \overrightarrow{PC} = \vec{0} \Longrightarrow (3x_1 + 2x_2 + x_3)\mathbf{i} + (3y_1 + 2y_2 + y_3)\mathbf{j} = 0$$

$$\Longrightarrow \begin{cases} 3x_1 + 2x_2 + x_3 = 0 \\ 3y_1 + 2y_2 + y_3 = 0 \end{cases} \Longrightarrow \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} : \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix} : \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = 3:2:1$$

$$\therefore \quad [\Delta PBC]:[\Delta PCA]:[\Delta PAB] = \frac{1}{2}\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} : \frac{1}{2}\begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix} : \frac{1}{2}\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = 3:2:1$$

Method 6

There is a one-to-one correspondence between vector and complex number.
For any vector $\vec{v} = x\mathbf{i} + y\mathbf{j}$, there is a complex number $z = x + yi$.

The problem then becomes:
Take P to be the origin of the complex plane and $A(z_1), B(z_2)$ and $C(z_3)$, and $3z_1 + 2z_2 + z_3 = 0$, find $[\Delta PBC]:[\Delta PCA]:[\Delta PAB]$, where $[\Delta PAB]$ represents the area of the ΔPAB.

Since $3z_1 + 2z_2 + z_3 = 0$, we have the conjugate equation $3\overline{z_1} + 2\overline{z_2} + \overline{z_3} = 0$.
Hence $(z_2\overline{z_3} - z_3\overline{z_2}):(z_3\overline{z_1} - z_1\overline{z_3}):(z_1\overline{z_2} - z_2\overline{z_1}) = 3:2:1$.

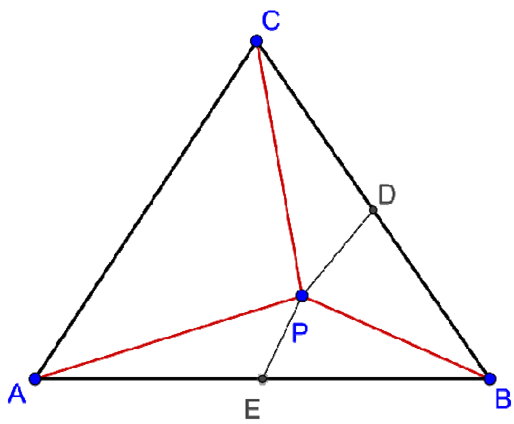
By the area formula in complex number, we have

$$[\Delta PBC]:[\Delta PCA]:[\Delta PAB] = \frac{i}{4}\begin{vmatrix} 1 & 1 & 1 \\ 0 & z_2 & z_3 \\ 0 & \overline{z_2} & \overline{z_3} \end{vmatrix} : \frac{i}{4}\begin{vmatrix} 1 & 1 & 1 \\ 0 & z_3 & z_1 \\ 0 & \overline{z_3} & \overline{z_1} \end{vmatrix} : \frac{i}{4}\begin{vmatrix} 1 & 1 & 1 \\ 0 & z_1 & z_2 \\ 0 & \overline{z_1} & \overline{z_2} \end{vmatrix}$$

$$= (z_2\overline{z_3} - z_3\overline{z_2}):(z_3\overline{z_1} - z_1\overline{z_3}):(z_1\overline{z_2} - z_2\overline{z_1})$$

$$= 3:2:1$$

(b) Place a coordinate system with A as origin and $\overrightarrow{AB}$ along x-axis.



Since $a\overrightarrow{PA} + b\overrightarrow{PB} + c\overrightarrow{PC} = \vec{0}$

Consider the y-cordinates,

$$-ay_P - by_P + c(y_c - y_P) = 0$$

$$y_C = \frac{c}{a+b+c} y_P$$

Hence, $[\Delta PAB] = \frac{c}{a+b+c}[\Delta ABC]$

If we set a coordinate system with B as origin and $\overrightarrow{BC}$ along x-axis, then: $[\Delta PBC] = \frac{a}{a+b+c}[\Delta ABC]$

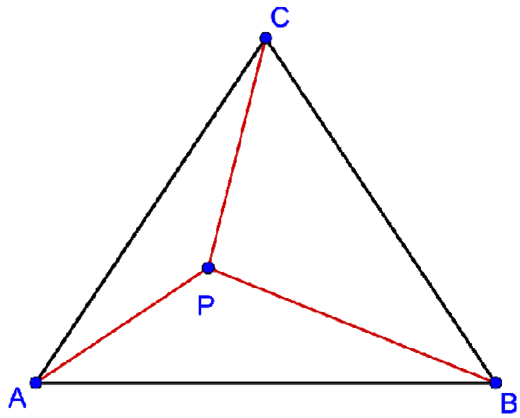
If we set a coordinate system with C as origin and $\overrightarrow{CA}$ along x-axis, then: $[\Delta PCA] = \frac{b}{a+b+c}[\Delta ABC]$

$$\therefore \quad [\Delta PBC]:[\Delta PCA]:[\Delta PAB] = \frac{a}{a+b+c}:\frac{b}{a+b+c}:\frac{c}{a+b+c} = a:b:c$$

(c) $\overrightarrow{PA} + 5\overrightarrow{PB} + 3\overrightarrow{PC} = \vec{0}$, by (**b**),

$$[\Delta PBC]:[\Delta PCA]:[\Delta PAB] = 1:5:3$$

$[\Delta PBC] = k,\ [\Delta PCA] = 5k,\ [\Delta PAB] = 3k$



Therefore, $[\Delta ABC] = k + 5k + 3k = 9k$

and $[ABPCA] = 3k + 5k = 8k$

$[\Delta ABC]:[ABPCA] = 9k:8k = 9:8$

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Method 6 of part (a) added on 10-3-16