**Discover and Verify**

The sum of the powers of the first n positive integers gives a list of interesting series:

$$1+2+…+n=\frac{n^{2}+n}{2}=\frac{n\left(n+1\right)}{2} …(1)$$

$$1^{2}+2^{2}+…+n^{2}=\frac{n^{3}}{3}+\frac{n^{2}}{2}+\frac{n}{6}=\frac{n\left(n+1\right)\left(2n+1\right)}{6} …(2)$$

$$1^{3}+2^{3}+…+n^{3}=\frac{n^{4}}{4}+\frac{n^{3}}{2}+\frac{n^{2}}{4}=\left[\frac{n\left(n+1\right)}{2}\right]^{2} …(3)$$

$$1^{4}+2^{4}+…+n^{4}=\frac{n^{5}}{5}+\frac{n^{4}}{2}+\frac{n^{3}}{3}-\frac{n}{30}=\frac{n\left(n+1\right)\left(2n+1\right)\left(3n^{2}+3n-1\right)}{30} …(4)$$

The pursuit of the coefficients in the formulas ended up in the rather old story called the Faulhaber’s theorem. The Johann Faulhaber’s formula expresses the sum of the k-th powers of the first n positive integers.

$$\sum\_{i=0}^{n}i^{k}=1^{k}+2^{k}+…+n^{k}$$

$$\sum\_{i=1}^{n}i^{k}=\frac{1}{k+1}\sum\_{j=1}^{k}\left(-1\right)^{j}\left(\begin{matrix}k+1\\j\end{matrix}\right)B\_{j}n^{k+1-j}$$

where $B\_{1}=-\frac{1}{2}$ and $B\_{j}$’s are Bernoulli numbers

The understanding and proof are remote for most secondary students and you may forget about it.

However, here is a mysterious pattern I came across for finding the coefficients you may enjoy.

First, study carefully the Coefficient Triangle below:

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  | **11** |  |  |  |  |  |  |
|  |  |  |  |  | **11** |  | **12** |  |  |  |  |  |
|  |  |  |  | **11** |  | **32** |  | **23** |  |  |  |  |
|  |  |  | **11** |  | **72** |  | **123** |  | **64** |  |  |  |
|  |  | **11** |  | **152** |  | **503** |  | **604** |  | **245** |  |  |
|  | **11** |  | **312** |  | **1303** |  | **3904** |  | **3655** |  | **1206** |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |

**Task 1 Discover how the numbers in the Coefficient Triangle are constructed.**

**Hints :** **1.** Study the subscripts in red. How to put in the numbers?

 **2.** How to find the numbers in black? Recall the Pascal triangle in Binomial Theorem, but here you need to do some multiplications.

**Answer for 2** Any number in black is the sum of the product of the number and its subscript directly a row above e.g. $365=60×4+24×5$.

If we write : $\sum\_{i=0}^{n}i^{k}=1^{k}+2^{k}+…+n^{k}$ and $C\begin{matrix}n\\r\end{matrix}$ be the Binomial coefficients, we get:

$$\sum\_{i=1}^{n}i^{0}=1C\begin{matrix}n\\1\end{matrix} …(5)$$

$$\sum\_{i=1}^{n}i^{1}=1C\begin{matrix}n\\1\end{matrix}+1C\begin{matrix}n\\2\end{matrix} …(6)$$

$$\sum\_{i=1}^{n}i^{2}=1C\begin{matrix}n\\1\end{matrix}+3C\begin{matrix}n\\2\end{matrix}+2C\begin{matrix}n\\3\end{matrix} …(7)$$

$$\sum\_{i=1}^{n}i^{3}=1C\begin{matrix}n\\1\end{matrix}+7C\begin{matrix}n\\2\end{matrix}+12C\begin{matrix}n\\3\end{matrix}+6C\begin{matrix}n\\4\end{matrix} …(8)$$

$$\sum\_{i=0}^{n}i^{4}=1C\begin{matrix}n\\1\end{matrix}+15C\begin{matrix}n\\2\end{matrix}+50C\begin{matrix}n\\3\end{matrix}+60C\begin{matrix}n\\4\end{matrix}+24C\begin{matrix}n\\5\end{matrix} …(9)$$

$$\sum\_{i=0}^{n}i^{5}=1C\begin{matrix}n\\1\end{matrix}+31C\begin{matrix}n\\2\end{matrix}+130C\begin{matrix}n\\3\end{matrix}+390C\begin{matrix}n\\4\end{matrix}+365C\begin{matrix}n\\5\end{matrix}+120C\begin{matrix}n\\6\end{matrix} …(10)$$

Note that the coefficients in the formulas (5) – (10) are just the numbers (in black) in the Coefficient Triangle above.

**Task 2 Checking**

Show that the equations are correct.

$$\left(7\right)=\left(2\right), \left(8\right)=\left(3\right), \left(9\right)=(4)$$

**Proof of**$ \left(9\right)=(4)$

 $1C\begin{matrix}n\\1\end{matrix}+15C\begin{matrix}n\\2\end{matrix}+50C\begin{matrix}n\\3\end{matrix}+60C\begin{matrix}n\\4\end{matrix}+24C\begin{matrix}n\\5\end{matrix}$

 $=\left(C\begin{matrix}n\\2\end{matrix}+C\begin{matrix}n\\1\end{matrix}\right)+14\left(C\begin{matrix}n\\3\end{matrix}+C\begin{matrix}n\\2\end{matrix}\right)+36\left(C\begin{matrix}n\\4\end{matrix}+C\begin{matrix}n\\3\end{matrix}\right)+24\left(C\begin{matrix}n\\5\end{matrix}+C\begin{matrix}n\\4\end{matrix}\right)$

 $=C\begin{matrix}n+1\\2\end{matrix}+14C\begin{matrix}n+1\\3\end{matrix}+36C\begin{matrix}n+1\\4\end{matrix}+24C\begin{matrix}n+1\\5\end{matrix}$

 $=\frac{\left(n+1\right)n}{2!}+14\frac{\left(n+1\right)n\left(n-1\right)}{3!}+36\frac{\left(n+1\right)n\left(n-1\right)\left(n-2\right)}{4!}+24\frac{\left(n+1\right)n\left(n-1\right)\left(n-2\right)\left(n-3\right)}{5!}$

 $=\frac{\left(n+1\right)n}{2}+7\frac{\left(n+1\right)n\left(n-1\right)}{3}+3\frac{\left(n+1\right)n\left(n-1\right)\left(n-2\right)}{2}+\frac{\left(n+1\right)n\left(n-1\right)\left(n-2\right)\left(n-3\right)}{5}$

 $=\frac{\left(n+1\right)n}{30}\left[15+70\left(n-1\right)+45\left(n-1\right)\left(n-2\right)+6\left(n-1\right)\left(n-2\right)\left(n-3\right)\right]$

 $=\frac{\left(n+1\right)n}{30}\left[6 n^{3}+9 n^{2}+n-1\right]=\frac{n\left(n+1\right)\left(2n+1\right)\left(3n^{2}+3n-1\right)}{30}$

The general proof that this pattern works well for equation (10) or beyond may not be easy and is not discuss here. However, readers may try the followings.

**Task 3 Verification**

**1.** Prove that equation (7) or (2) is correct by Mathematical Induction.

**2.** Prove that equation (8) or (3) is correct by considering the identity:

$$i^{2}\left(i+1\right)^{2}-\left(i-1\right)^{2}i^{2}=4i^{3}$$

**3.** Prove that equation (9) or (4) is correct by considering the identity:

 $\left(i+1\right)^{5}-i^{5}=5i^{4}+10i^{3}+10i^{2}+5i+1$

**Verification Answers**

**1.** Let $P\left(n\right): \sum\_{i=1}^{n}i^{2}=1C\begin{matrix}n\\1\end{matrix}+3C\begin{matrix}n\\2\end{matrix}+2C\begin{matrix}n\\3\end{matrix}$

 $P\left(1\right): 1^{2}=1C\begin{matrix}1\\1\end{matrix}, P\left(2\right): 1^{2}+2^{2}=5=1×\frac{2×1}{1!}+3×\frac{2×1}{2!}=1C\begin{matrix}2\\1\end{matrix}+3C\begin{matrix}2\\2\end{matrix}$

 $P\left(3\right): 1^{2}+2^{2}+3^{2}=14=1×\frac{3}{1!}+3×\frac{3×2}{2!}+2×\frac{3×2×1}{3!}$

 are correct.

 Assume $P\left(k\right): \sum\_{i=1}^{k}i^{2}=1C\begin{matrix}k\\1\end{matrix}+3C\begin{matrix}k\\2\end{matrix}+2C\begin{matrix}k\\3\end{matrix}$ is true for some $kϵN$.

 For $P\left(k+1\right): $

 $\sum\_{i=1}^{k+1}i^{2}=\sum\_{i=1}^{k}i^{2}+\left(k+1\right)^{2}$

 $=1C\begin{matrix}k\\1\end{matrix}+3C\begin{matrix}k\\2\end{matrix}+2C\begin{matrix}k\\3\end{matrix}+\left(k+1\right)^{2}$

 $=1\left(C\begin{matrix}k+1\\1\end{matrix}-C\begin{matrix}k\\0\end{matrix}\right)+3\left(C\begin{matrix}k+1\\2\end{matrix}-C\begin{matrix}k\\1\end{matrix}\right)+2\left(C\begin{matrix}k+1\\3\end{matrix}-C\begin{matrix}k\\2\end{matrix}\right)+\left(k+1\right)^{2}$

 $=1C\begin{matrix}k+1\\1\end{matrix}+3C\begin{matrix}k+1\\2\end{matrix}+2C\begin{matrix}k+2\\3\end{matrix}-1-3k-2×\frac{k\left(k-1\right)}{2!}+\left(k+1\right)^{2}$

 $=1C\begin{matrix}k+1\\1\end{matrix}+3C\begin{matrix}k+1\\2\end{matrix}+2C\begin{matrix}k+2\\3\end{matrix}$

 $∴ P\left(k+1\right) is true. $

 By the Principle of Mathematical Induction, $P\left(n\right)$ is true for all $nϵN$.

**2.** We prove equation (3) here.

 Put $i=n, \left(n-1\right), …,3,2,1$ in the identity: $i^{2}\left(i+1\right)^{2}-\left(i-1\right)^{2}i^{2}=4i^{3}$.

$$n^{2}\left(n+1\right)^{2}-\left(n-1\right)^{2}n^{2}=4n^{3}$$

$$\left(n-1\right)^{2}n^{2}-\left(n-2\right)^{2}\left(n-1\right)^{2}=4\left(n-1\right)^{3}$$

 $…$

$$3^{2}4^{2}-2^{2}3^{2}=4×3^{3}$$

$$2^{2}3^{2}-1^{2}2^{2}=4×2^{3}$$

$$1^{2}2^{2}-0^{2}1^{2}=4×1^{3}$$

 Adding the above identities, we have:

$$n^{2}\left(n+1\right)^{2}=4×\sum\_{i=1}^{n}i^{3}$$

$$\sum\_{i=1}^{n}i^{3}=\left[\frac{n\left(n+1\right)}{2}\right]^{2}$$

**3.** We prove equation (4) here.

 Since $\left(i+1\right)^{5}-i^{5}=5i^{4}+10i^{3}+10i^{2}+5i+1$, taking sum from $i=1 to n,$

 $\sum\_{i=1}^{n}\left(i+1\right)^{5}-\sum\_{i=1}^{n}i^{5}=5\sum\_{i=1}^{n}i^{4}+10\sum\_{i=1}^{n}i^{3}+10\sum\_{i=1}^{n}i^{2}+5\sum\_{i=1}^{n}i+\sum\_{i=1}^{n}1$

 $\left(n+1\right)^{5}-1^{5}=5\sum\_{i=1}^{n}i^{4}+10\left[\frac{n\left(n+1\right)}{2}\right]^{2}+10\left[\frac{n\left(n+1\right)\left(2n+1\right)}{6}\right]+5\left[\frac{n\left(n+1\right)}{2}\right]+n$

 $n^{5}+5 n^{4}+10 n^{3}+10 n^{2}+4 n=5\sum\_{i=1}^{n}i^{4}+10\left[\frac{n\left(n+1\right)}{2}\right]^{2}+10\left[\frac{n\left(n+1\right)\left(2n+1\right)}{6}\right]+5\left[\frac{n\left(n+1\right)}{2}\right]+n$

 $5\sum\_{i=1}^{n}i^{4}=n^{5}+5 n^{4}+10 n^{3}+10 n^{2}+4 n-\frac{5}{2}\left[n\left(n+1\right)\right]^{2}-\frac{10}{3}n\left(n+1\right)\left(2n+1\right)-\frac{5}{2}n\left(n+1\right)$

$$5\sum\_{i=1}^{n}i^{4}=n \left(n+1\right) \left(n+2\right) \left(n^{2}+2 n+2\right)-\frac{5}{2}\left[n\left(n+1\right)\right]^{2}-\frac{10}{3}n\left(n+1\right)\left(2n+1\right)-\frac{5}{2}n\left(n+1\right)$$

$$30\sum\_{i=1}^{n}i^{4}=n\left(n+1\right)\left\{6\left(n+2\right) \left(n^{2}+2 n+2\right)-15n\left(n+1\right)-20\left(2n+1\right)-15\right\}$$

$$30\sum\_{i=1}^{n}i^{4}=n\left(n+1\right)\left\{6 n^{3}+9 n^{2}-19 n-11\right\}$$

$$30\sum\_{i=1}^{n}i^{4}=n\left(n+1\right)\left(2 n+1\right) \left(3 n^{2}+3 n-11\right)$$

$$\sum\_{i=1}^{n}i^{4}=\frac{n\left(n+1\right)\left(2n+1\right)\left(3n^{2}+3n-1\right)}{30}$$

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