**Discover and Verify**

The sum of the powers of the first n positive integers gives a list of interesting series:

The pursuit of the coefficients in the formulas ended up in the rather old story called the Faulhaber’s theorem. The Johann Faulhaber’s formula expresses the sum of the k-th powers of the first n positive integers.

where and ’s are Bernoulli numbers

The understanding and proof are remote for most secondary students and you may forget about it.

However, here is a mysterious pattern I came across for finding the coefficients you may enjoy.

First, study carefully the Coefficient Triangle below:

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  | **11** |  |  |  |  |  |  |
|  |  |  |  |  | **11** |  | **12** |  |  |  |  |  |
|  |  |  |  | **11** |  | **32** |  | **23** |  |  |  |  |
|  |  |  | **11** |  | **72** |  | **123** |  | **64** |  |  |  |
|  |  | **11** |  | **152** |  | **503** |  | **604** |  | **245** |  |  |
|  | **11** |  | **312** |  | **1303** |  | **3904** |  | **3655** |  | **1206** |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |

**Task 1 Discover how the numbers in the Coefficient Triangle are constructed.**

**Hints :** **1.** Study the subscripts in red. How to put in the numbers?

**2.** How to find the numbers in black? Recall the Pascal triangle in Binomial Theorem, but here you need to do some multiplications.

**Answer for 2** Any number in black is the sum of the product of the number and its subscript directly a row above e.g. .

If we write : and be the Binomial coefficients, we get:

Note that the coefficients in the formulas (5) – (10) are just the numbers (in black) in the Coefficient Triangle above.

**Task 2 Checking**

Show that the equations are correct.

**Proof of**

The general proof that this pattern works well for equation (10) or beyond may not be easy and is not discuss here. However, readers may try the followings.

**Task 3 Verification**

**1.** Prove that equation (7) or (2) is correct by Mathematical Induction.

**2.** Prove that equation (8) or (3) is correct by considering the identity:

**3.** Prove that equation (9) or (4) is correct by considering the identity:

**Verification Answers**

**1.** Let

are correct.

Assume is true for some .

For

By the Principle of Mathematical Induction, is true for all .

**2.** We prove equation (3) here.

Put in the identity: .

Adding the above identities, we have:

**3.** We prove equation (4) here.

Since , taking sum from

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