Discover and Verify

The sum of the powers of the first n positive integers gives a list of interesting series:

$$1+2+\cdots+n=\frac{n^2+n}{2}=\frac{n(n+1)}{2} \quad \ldots(1)$$

$$1^2+2^2+\cdots+n^2=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}=\frac{n(n+1)(2n+1)}{6} \quad \ldots(2)$$

$$1^3+2^3+\cdots+n^3=\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}=\left[\frac{n(n+1)}{2}\right]^2 \quad \ldots(3)$$

$$1^4+2^4+\cdots+n^4=\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \quad \ldots(4)$$

The pursuit of the coefficients in the formulas ended up in the rather old story called the Faulhaber's theorem. The Johann Faulhaber's formula expresses the sum of the k-th powers of the first n positive integers.

$$\sum_{i=0}^{n} i^k = 1^k+2^k+\cdots+n^k$$

$$\sum_{i=1}^{n} i^k = \frac{1}{k+1}\sum_{j=1}^{k}(-1)^j\binom{k+1}{j}B_j n^{k+1-j}$$

where $B_1=-\frac{1}{2}$ and B_j's are Bernoulli numbers

The understanding and proof are remote for most secondary students and you may forget about it.

However, here is a mysterious pattern I came across for finding the coefficients you may enjoy.

First, study carefully the Coefficient Triangle below:

						1_1						
					1_1		1_2					
				1_1		3_2		2_3				
			1_1		7_2		12_3		6_4			
		1_1		15_2		50_3		60_4		24_5		
	1_1		31_2		130_3		390_4		365_5		120_6	

Task 1 Discover how the numbers in the Coefficient Triangle are constructed.

Hints :

1. Study the subscripts in red. How to put in the numbers?
2. How to find the numbers in black? Recall the Pascal triangle in Binomial Theorem, but here you need to do some multiplications.

Answer for 2 Any number in black is the sum of the product of the number and its subscript directly a row above e.g. $365 = 60 \times 4 + 24 \times 5$.

If we write : $\sum_{i=0}^{n} i^k = 1^k + 2^k + \cdots + n^k$ and C_r^n be the Binomial coefficients, we get:

$$\sum_{i=1}^{n} i^0 = 1C_1^n \quad \ldots(5)$$

$$\sum_{i=1}^{n} i^1 = 1C_1^n + 1C_2^n \quad \ldots(6)$$

$$\sum_{i=1}^{n} i^2 = 1C_1^n + 3C_2^n + 2C_3^n \quad \ldots(7)$$

$$\sum_{i=1}^{n} i^3 = 1C_1^n + 7C_2^n + 12C_3^n + 6C_4^n \quad \ldots(8)$$

$$\sum_{i=0}^{n} i^4 = 1C_1^n + 15C_2^n + 50C_3^n + 60C_4^n + 24C_5^n \quad \ldots(9)$$

$$\sum_{i=0}^{n} i^5 = 1C_1^n + 31C_2^n + 130C_3^n + 390C_4^n + 365C_5^n + 120C_6^n \quad \ldots(10)$$

Note that the coefficients in the formulas (5) – (10) are just the numbers (in black) in the Coefficient Triangle above.

Task 2 Checking

Show that the equations are correct.

$(7) = (2), \qquad (8) = (3), \qquad (9) = (4)$

Proof of $(9) = (4)$

$$1C_1^n + 15C_2^n + 50C_3^n + 60C_4^n + 24C_5^n$$

$$= \left(C_2^n + C_1^n\right) + 14\left(C_3^n + C_2^n\right) + 36\left(C_4^n + C_3^n\right) + 24\left(C_5^n + C_4^n\right)$$

$$= C_2^{n+1} + 14C_3^{n+1} + 36C_4^{n+1} + 24C_5^{n+1}$$

$$= \frac{(n+1)n}{2!} + 14\frac{(n+1)n(n-1)}{3!} + 36\frac{(n+1)n(n-1)(n-2)}{4!} + 24\frac{(n+1)n(n-1)(n-2)(n-3)}{5!}$$

$$= \frac{(n+1)n}{2} + 7\frac{(n+1)n(n-1)}{3} + 3\frac{(n+1)n(n-1)(n-2)}{2} + \frac{(n+1)n(n-1)(n-2)(n-3)}{5}$$

$$= \frac{(n+1)n}{30}[15 + 70(n-1) + 45(n-1)(n-2) + 6(n-1)(n-2)(n-3)]$$

$$= \frac{(n+1)n}{30}[6\,n^3 + 9\,n^2 + n - 1] = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

The general proof that this pattern works well for equation (10) or beyond may not be easy and is not discuss here. However, readers may try the followings.

Task 3 Verification

1. Prove that equation (7) or (2) is correct by Mathematical Induction.

2. Prove that equation (8) or (3) is correct by considering the identity:

$$i^2(i+1)^2 - (i-1)^2 i^2 = 4i^3$$

3. Prove that equation (9) or (4) is correct by considering the identity:

$$(i+1)^5 - i^5 = 5i^4 + 10i^3 + 10i^2 + 5i + 1$$

Verification Answers

1. Let $P(n)$: $\sum_{i=1}^{n} i^2 = 1C_1^n + 3C_2^n + 2C_3^n$

$P(1)$: $1^2 = 1C_1^1$, $P(2)$: $1^2 + 2^2 = 5 = 1 \times \frac{2\times1}{1!} + 3 \times \frac{2\times1}{2!} = 1C_1^2 + 3C_2^2$

$P(3)$: $1^2 + 2^2 + 3^2 = 14 = 1 \times \frac{3}{1!} + 3 \times \frac{3\times2}{2!} + 2 \times \frac{3\times2\times1}{3!}$

are correct.

Assume $P(k)$: $\sum_{i=1}^{k} i^2 = 1C_1^k + 3C_2^k + 2C_3^k$ is true for some $k \in \mathbf{N}$.

For $P(k+1)$:

$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$

$= 1C_1^k + 3C_2^k + 2C_3^k + (k+1)^2$

$= 1\left(C_1^{k+1} - C_0^k\right) + 3\left(C_2^{k+1} - C_1^k\right) + 2\left(C_3^{k+1} - C_2^k\right) + (k+1)^2$

$= 1C_1^{k+1} + 3C_2^{k+1} + 2C_3^{k+2} - 1 - 3k - 2 \times \frac{k(k-1)}{2!} + (k+1)^2$

$= 1C_1^{k+1} + 3C_2^{k+1} + 2C_3^{k+2}$

$\therefore$ $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for all $n \epsilon \mathbf{N}$.

2. We prove equation (3) here.

Put $i = n, (n-1), \ldots, 3, 2, 1$ in the identity: $i^2(i+1)^2 - (i-1)^2 i^2 = 4i^3$.

$$n^2(n+1)^2 - (n-1)^2 n^2 = 4n^3$$
$$(n-1)^2 n^2 - (n-2)^2 (n-1)^2 = 4(n-1)^3$$
$$\ldots$$
$$3^2 4^2 - 2^2 3^2 = 4 \times 3^3$$
$$2^2 3^2 - 1^2 2^2 = 4 \times 2^3$$
$$1^2 2^2 - 0^2 1^2 = 4 \times 1^3$$

Adding the above identities, we have:

$$n^2(n+1)^2 = 4 \times \sum_{i=1}^{n} i^3$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

3. We prove equation (4) here.

Since $(i+1)^5 - i^5 = 5i^4 + 10i^3 + 10i^2 + 5i + 1$, taking sum from $i = 1$ to n,

$\sum_{i=1}^{n} (i+1)^5 - \sum_{i=1}^{n} i^5 = 5\sum_{i=1}^{n} i^4 + 10\sum_{i=1}^{n} i^3 + 10\sum_{i=1}^{n} i^2 + 5\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$

$(n+1)^5 - 1^5 = 5\sum_{i=1}^{n} i^4 + 10\left[\frac{n(n+1)}{2}\right]^2 + 10\left[\frac{n(n+1)(2n+1)}{6}\right] + 5\left[\frac{n(n+1)}{2}\right] + n$

$n^5 + 5\,n^4 + 10\,n^3 + 10\,n^2 + 4\,n = 5\sum_{i=1}^{n} i^4 + 10\left[\frac{n(n+1)}{2}\right]^2 + 10\left[\frac{n(n+1)(2n+1)}{6}\right] + 5\left[\frac{n(n+1)}{2}\right] + n$

$$5\sum_{i=1}^{n} i^4 = n^5 + 5\,n^4 + 10\,n^3 + 10\,n^2 + 4\,n - \frac{5}{2}[n(n+1)]^2 - \frac{10}{3}n(n+1)(2n+1) - \frac{5}{2}n(n+1)$$

$$5\sum_{i=1}^{n} i^4 = n\,(n+1)\,(n+2)\,(n^2 + 2\,n + 2) - \frac{5}{2}[n(n+1)]^2 - \frac{10}{3}n(n+1)(2n+1) - \frac{5}{2}n(n+1)$$

$$30\sum_{i=1}^{n} i^4 = n(n+1)\{6(n+2)\,(n^2 + 2\,n + 2) - 15n(n+1) - 20(2n+1) - 15\}$$

$$30\sum_{i=1}^{n} i^4 = n(n+1)\{6\,n^3 + 9\,n^2 - 19\,n - 11\}$$

$$30\sum_{i=1}^{n} i^4 = n(n+1)(2\,n+1)\,(3\,n^2 + 3\,n - 11)$$

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$$

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